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Finite Dimensional Attractor for a One-Phase Stefan Problem with Kinetics

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For a one-phase free-boundary problem with kinetics, which is known to generate a rich dynamics, we study evolution of the infinitesimal volume along the trajectories in the attractor. We demonstrate that for sufficiently large m that is defined solely by the properties of the kinetics function the m-dimensional volume decays exponentially. This property combined with the uniform differentiability of the semigroup leads to the conclusion that the Hausdorff dimension of the attractor is finite.

KEY WORDS: Free boundaries; Hausdorff dimension.

1. INTRODUCTION

In this paper we study structure of the attractor for a modified one-phase Stefan problem and present an argument that its Hausdorff dimension is finite. The results are motivated by DNS and in particular by the numerical experiments described in [Frankel *et al.* (1994), Frankel and Roytburd (1994)]. Indeed, in [Frankel and Roytburd (1994)] we demonstrated numerically that the free-boundary problem described below develops a remarkable variety of complex thermokinetic oscillations as, in a certain parameter range, the basic traveling wave solution becomes unstable. As the governing parameter is varied, the system exhibits dynamical patterns that include a Hopf bifurcation, period doubling cascades leading to

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chaotic pulsations, a Shilnikov–Hopf bifurcation etc. Most of these patterns are well-known for the finite dimensional dynamical systems.

At the same time, in [Frankel and Roytburd (1995)], [Frankel *et al.* (2000)] we demonstrated that a 3×3 system of ODEs obtained as a pseudo-spectral approximation to the free-boundary problem exhibits dynamics that mimic that of the infinite dimensional system to an amazing degree. These observations led us to the conjecture that the asymptotic dynamics of the free-boundary problem should be finite dimensional.

The free-boundary problem is formulated as follows

$$u_t = u_{xx} - \gamma u, \qquad -\infty < x < s(t), \tag{1.1}$$

$$(\partial u/\partial x)|_{x=s(t)} = -V(t), \qquad g(u|_{x=s(t)}) = V(t), \tag{1.2}$$

$$u(x, 0) = u^0(x). \tag{1.3}$$

Here $u(x, t)$ is the temperature, the damping term is due to the volumetric heat losses, $\gamma \geqslant 0$; the two boundary conditions overdetermine the problem and allow one to find the free boundary whose position is $s(t)$ and velocity, $V(t) = \dot{s}(t)$. Physical motivations for the model are briefly discussed in [Frankel and Roytburd (2002a)].

The free-boundary problem (1.1–1.3) arises naturally as a mathematical model for a variety of exothermic phase transition type processes, such as condensed phase combustion [Matkowsky and Sivashinsky (1978)] also known as self-sustained high-temperature synthesis or SHS [Munir and Anselmi-Tamburini (1989)], solidification with undercooling [Langer (1987)], laser induced evaporation [Gol'berg and Tribelskii (1985)], rapid crystallization in thin films [van Saarloos and Weeks (1984)] etc. These processes are characterized by production of heat at the interface, and their dynamics is determined by the feedback mechanism between the heat release due to the kinetics $g(u|_{x=s(t)})$ and the heat dissipation by the medium. The first boundary condition in (1.2) (the Stefan boundary condition) expresses the balance between the heat produced at the free boundary and its diffusion by the adjacent medium. As the problem (1.1–1.3) describes propagation of the phase transition front, the second boundary condition in (1.2) is a manifestation of the nonequilibrium nature of the transition; its analog for the classical Stefan problem is just $u|_{x=s(t)} = 0$.

There is a substantial literature that treats analytical aspects of the initial-boundary value problem for different sharp-interface models related to the model in (1.1–1.3), see [Luckhaus (1990), Radkevich (1991), Chen and Reitich (1992), Yin (1993), Din *et al.* (1997)]. These works are concerned with basic issues of mostly local in time existence. We also note recent papers [Brauner *et al.* (2000)], [Brauner and Lunardi (2000)], where the authors study dynamical behavior of solutions of a related problem. In particular they consider perturbations of traveling-wave initial data and investigate their instability and bifurcations. In contrast, the principal focus of the present paper is in asymptotic dynamics for a wide range of initial data and parametric regimes.

The rest of the paper is organized as follows. In Section 2 we present some minimal background information from [Frankel and Roytburd (2002a)] on global in time existence of classical solutions, and present a brief argument that any bounded set has a compact attractor. In Section 3 we study evolution of the infinitesimal volume element along the trajectories in the attractor. We demonstrate that for sufficiently large m that is defined solely by the properties of the kinetics function, the m-dimensional volume decays exponentially. In the last section we introduce a notion of a weak solution and prove that the evolution semigroup is uniformly differentiable on a dense subspace of weak solutions. Combined with the estimate for the linearized evolution of the infinitesimal volume, this leads to the conclusion that the Hausdorff dimension of the attractor is finite.

2. EXISTENCE OF CLASSICAL SOLUTIONS, AND A COMPACT ATTRACTOR

In this section for readers convenience we briefly describe the necessary prior results [Frankel and Roytburd (2002a)] establishing well-posedness of the free-boundary problem and existence of a compact attractor.

A short-time classical solution of the free-boundary problem is sought in the form of a superposition of heat potentials,

$$u(x, t) = \int_0^t \tilde{G}(x, s(\tau), t-\tau)\, \varphi(\tau)\, d\tau + \int_{-\infty}^0 \tilde{G}(x, \xi, t)\, u^0(\xi)\, d\xi, \qquad (2.1)$$

where $\tilde{G} = e^{-\gamma(t-\tau)}G$ is the fundamental solution of the heat equation with damping, G is the Gaussian kernel

$$G(x, \xi, t-\tau) \equiv G(x, t, \xi, \tau) = \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\}[4\pi(t-\tau)]^{-1/2}.$$

The density of the single layer potential φ and the front position $s(t)$ are to be determined. It is not hard to see that φ must have a $1/\sqrt{t}$ singularity at 0. In spite of this singularity it can be proved that the single layer potential possesses the standard jump property (which of course is well-known for φ

continuous). The jump condition together with the kinetic boundary condition result in the following system of integral equations

$$u(s(t),t)=g^{-1}(V(t))=\int_0^t \tilde{G}(s(t),s(\tau),t-\tau)\,\varphi(\tau)\,d\tau \\ +\int_{-\infty}^0 \tilde{G}(s(t),\xi,t)\,u^0(\xi)\,d\xi \tag{2.2}$$

$$u_x(s(t),t)=-V(t)=\varphi/2-\int_0^t \tilde{G}_\xi(s(t),s(\tau),t-\tau)\,\varphi(\tau)\,d\tau \\ -\int_{-\infty}^0 \tilde{G}_\xi(s(t),\xi,t)\,u^0(\xi)\,d\xi. \tag{2.3}$$

The equations should be supplemented by the compatibility initial conditions: $V(0)=g(u^0(0))$, $\lim_{t\to 0}\sqrt{t}\,\varphi(t)=u^0(0)/\sqrt{\pi}$.

Theorem 1. *Let $g<0$ be continuously differentiable, monotone decreasing, $u^0\in C(-\infty,0]$, $u^0>0$. Then the problem in (2.2)–(2.3) has a unique solution $\{V,\varphi\}$ such that V and $\sqrt{t}\,\varphi(t)$ are continuous on $[0,\sigma]$ for some $\sigma>0$, where σ depends only on $\sup u^0$. A solution to the free boundary problem is determined by V, φ via the representation (2.1).*

The theorem differs from our earlier result as well as from other existence results, [Chen and Reitich (1992), Radkevich (1991), Yin (1993)] in not requiring any smoothness from the initial data. The proof is based on a contraction argument for sufficiently small time interval σ.

The proof of global existence is conditioned upon an extra requirement on the kinetic function:

$$-V_0\leqslant g\leqslant -v_0<0. \tag{2.4}$$

We note that the lower bound is satisfied for the standard Arrhenius kinetics (actually the result holds even if g has a sublinear growth). The upper bound v_0 corresponds to the "ignition velocity:" the model is valid only for moving fronts; this bound can be dropped in the presence of heat losses, $\gamma>0$. Global existence is guaranteed by the following a priori estimate:

Theorem 2. *Let u be a solution of the free-boundary problem, then the interface temperature $U(\tau)=u(s(\tau),\tau)$ is uniformly bounded $|U(\tau)|\leqslant R_{fb}+2e^{-\gamma t}\,\|u^0\|_{C(-\infty,0)}$, where the constant $R_{fb}=g^{-1}(-V_0/2)\,V_0/(v_0+\sqrt{\gamma})$ is totally determined by the kinetic function. By the maximum principle, u is also uniformly bounded.*

In addition to the uniform estimate for the solution we have also the following estimate for the derivative u_x which is necessary for the proof of compactness of the attractor.

Theorem 3. *Consider the ball* $\|u^0\| \leqslant R$. *There exists* $\sigma > 0$ *depending on* R *such that for any fixed* t, $0 < t \leqslant \sigma$, *the derivative of the solutions of the free boundary problem with the initial data from the ball is uniformly bounded. More specifically*

$$|u_x(x,t)| \leqslant \frac{C}{1+|s(t)-x|}$$

where C *is determined by* R *and* t. *For all* $t \geqslant t_0$, *where* $t_0 \leqslant \sigma$ *the derivative is uniformly bounded:* $|u_x(x,t)| \leqslant C$.

The estimate in Theorem 2 is based on the following representation for the solution:

$$u(x,t) = \int_0^t \tilde{G}(x, s(\tau), t-\tau)[-V(\tau)+U(\tau)\,V(\tau)]\,d\tau$$

$$-\int_0^t \frac{\partial \tilde{G}}{\partial \xi}(x, s(\tau), t-\tau)\,u(\tau)\,d\tau + \int_{-\infty}^0 \tilde{G}(x,\xi,t)\,u^0(\xi)\,d\xi \tag{2.5}$$

which is obtained by integrating Green's identity over the domain $\xi < s(\tau)$, $0 < \tau < t$. Since both U and V are determined by the initial conditions, the representation can be thought of as the time evolution of the initial temperature distribution u^0: $u(t) = T(t)\,u^0$. We understand the evolution as taking place for the functions on the fixed interval $(-\infty, 0)$. This is equivalent to the introduction of the moving coordinate system attached to the free boundary $x' = x - s(t)$. We split the semigroup operator T into two parts: the contribution of the free boundary and that of the initial data

$$T_1(t)\,u^0 = \int_0^t \tilde{G}(x, s(\tau), t-\tau)[-V(\tau)+U(\tau)\,V(\tau)]\,d\tau$$

$$-\int_0^t \frac{\partial \tilde{G}}{\partial \xi}(x, s(\tau), t-\tau)\,U(\tau)\,d\tau \tag{2.6}$$

$$T_2(t)\,u^0 = e^{-\gamma t} \int_{-\infty}^0 G(x', \xi - s(t), t)\,u^0(\xi)\,d\xi. \tag{2.7}$$

Another basic ingredient is exponential decay of the contribution from T_1:

$$|T_1(t)\, u^0(x')| \leqslant \frac{\sqrt{2}\, V_0(R_{fb}+2e^{-\gamma t}N+1)}{v_0}\, e^{-v_0\,|x'|/4}. \tag{2.8}$$

As a basic metric space we choose a ball in the space $C(-\infty, 0]$:

$$X=\{u \in C(-\infty, 0];\ \|u\|=\sup |u(x')| \leqslant N\}$$

where the radius N is large enough (it suffices to take $N > R_{fb}+2R_{\text{abs}}$ where R_{abs} is the radius of the absorbing ball which is estimated in the following proposition). Note that by Lemma 2, the evolution of any ball B_R of radius $R \leqslant (N-R_{fb})/2$ stays in X for all time.

Based on the above estimates we can prove

Proposition 4.

(i) *The semigroup T_2 is uniformly exponentially contracting:*

$$r_X(t)=\sup_{u^0 \in X} \|T_2(t)\, u^2\| \leqslant C \exp(-\gamma t)\, N \qquad \text{as} \quad t \to \infty.$$

(ii) *There exists a constant, R_{abs}, totally determined by the kinetics such that any ball $B_a=\{u \in X;\ \|u\| \leqslant R_{\text{abs}}\}$ is an absorbing set for the ball B_R with respect to the evolution by T_1 (and T).*

(iii) *The semigroup $T_1(t)$ is uniformly compact: there exists $t_0>0$ such that $\bigcup_{t \geqslant t_0} T_1(t)\, X$ is relatively compact in X.*

The properties of the evolution operator $T(t)$ described in the above propositions allow us to apply the abstract general result (see, for example, [Temam (1988), Chap. I]) that in our situation can be stated as follows:

Theorem 5. *The continuous semigroup $T(t)$, $T(t)=T_1(t)+T_2(t)$ with $T_1(t)$ uniformly compact and $T_2(t)$ uniformly contracting has the following properties: the ω-limit set A of the absorbing set B_a is a compact attractor for the metric space X; A is the maximal attractor in X and it is connected.*

3. EVOLUTION OF THE VOLUME ELEMENTS ON THE ATTRACTOR

Now we are ready to present the main result of the paper which is an estimate on the Hausdorff dimension of the attractor. In order to obtain the estimate we study evolution of the infinitesimal volume along the trajectories in the attractor. We demonstrate that for sufficiently large m that is defined solely by the properties of the kinetics function the m-dimensional volume decays exponentially. This property combined with the compactness suggests that the Hausdorff dimension of the attractor for the solutions of the free boundary problem is no larger than m. In the arguments regarding the Hausdorff dimension of the attractor we follow quite closely the ideas outlined in [Temam (1988)].

For simplicity of presentation, in all computations of the present section we set the heat losses to 0, $\gamma = 0$; we remark at the end of the section which (trivial) changes should be made for the case $\gamma > 0$. First we reformulate the problem in the coordinate frame attached to the free boundary, $\tilde{x} = x - s(t)$ as follows

$$\begin{gathered} u_t = u_{xx} + v(t)\, u_x = F(u), \qquad -\infty < x < 0, \\ g(u|_{x=0}) = v(t), \qquad (\partial u/\partial x)|_{x=0} = -v(t), \\ u(x, 0) = u^0(x). \end{gathered} \tag{3.1}$$

(Tildes have been omitted)

For the analysis to follow we need to introduce two weighted Hilbert spaces:

$$\begin{aligned} H_\alpha &= \{ f \mid e^{-\alpha x/2} f \in L_2(-\infty, 0) \}, \\ H_\alpha^1 &= \{ f \mid e^{-\alpha x/2} f \in H^1(-\infty, 0) \}, \qquad \alpha \geqslant 0. \end{aligned}$$

The choice of the weight is dictated by the spectral properties of the linearized problem. An appropriate choice of the parameter α allows one to make negative the spectrum of the linearization about any solution from the attractor.

Remark 1. In view of the exponential decay at $-\infty$ of functions in the attractor, it can be easily demonstrated that A is a relatively compact set in the H_α-metric for sufficiently small α.

Let $\{U(\cdot, t), V(t)\}$ be an orbit in the attractor. Let us consider the linearization of the problem (3.1) about U:

$$w_t = w_{xx} + w_x V - w(0, t)\, U_x / v(V(t)) = F'(U, V)\, w \tag{3.2}$$

$$w(0, t) - w_x(0, t)\, v(V(t)) = 0, \tag{3.3}$$

$$w(x, 0) = w_0(x) \tag{3.4}$$

where $v(V) = -(g^{-1})'(V)$. We require $v(V)$ to be positive and bounded from below. This condition again mimics the behavior of the Arrhenius kinetics. In Eq. (3.2) we have replaced the velocity perturbation $v(t)$ in the term $v(t)\,U_x$ of the linearization by the perturbation of the temperature $w(0, t)/v(V(t))$ that arises from the linearization of the kinetic boundary condition in (3.1).

The linearized problem represents the first variation of problem (3.1). It is possible to show that the linearized problem is locally well-posed (we only need local existence for the volume estimate) in the following sense:

Theorem 6. *For any* $w_0 \in H_\alpha = L^2_\alpha(-\infty, 0)$ *there exists a unique solution* w *of (3.2–3.4) such that* $w \in L^2(0, T; \Xi(t)) \cap C([0, T]; H_\alpha)$ *where* $\Xi(t) = \{f \in H^1_\alpha,\ f(0) - f_x(0)\, v(V(t)) = 0\}$.

Proof. This linear problem is somewhat nonstandard as it contains a nonlocal term (projection) $w(0, t)$. Nonetheless it can be handled as follows. Consider first the problem (3.2–3.4) with a source, and zero initial conditions

$$\tilde{w}_t = \tilde{w}_{xx} + \tilde{w}_x V + \mathscr{F}(x, t)$$

$$\tilde{w}(0, t) - \tilde{w}_x(0, t)\, v(V(t)) = 0,$$

$$\tilde{w}(x, 0) = 0,$$

and let $\mathfrak{L}$ be its solution operator: $\tilde{w} = \mathfrak{L}\mathscr{F}(x, t)$. Existence of unique global solutions for such problems is guaranteed by the general theory of linear parabolic equations.

We regard a solution of (3.2–3.4) as a superposition of an appropriate $\tilde{w}$ and of $W(x, t)$ which solves the *homogeneous* problem with the initial condition $w_0(x)$. Then, with the nonlocal term viewed as a source, on the boundary one obtains an equation for $w(0, t)$:

$$\mathfrak{L}[-w(0, t)\, U_x/v(V(t)) + W(x, t)]_{x=0} = w(0, t) + W(0, t). \tag{3.5}$$

It is not difficult to show that the above equation is uniquely solvable as an integral equation with a sufficiently regular kernel. Thus, the source term is found and, consequently the problem (3.2–3.4) can be solved. □

We are now ready to estimate the evolution of the volume element. To this end we need to estimate the trace of the finite dimensional projections of the generator of the linear semigroup. Let $\{\xi_1, \ldots, \xi_m\}$ be m elements of H_α and let $\{w_1, \ldots, w_m\}$ be the corresponding solutions of the linearized problem. Then it can be shown that the volume element spanned by $\{\xi_1,\ldots,\xi_m\}$ evolves accordingly to the formula

$$|w_1(t)\wedge\cdots\wedge w_m(t)| = |\xi_1\wedge\cdots\wedge\xi_m| \exp\int_0^t Tr[F'(U(\tau),V(\tau))\circ Q_m(\tau)]\,d\tau,$$

where $Q_m(\tau)=Q_m(\tau,U,V;\xi_1,\ldots,\xi_m)$ is the projector in H_α onto the space spanned by $\Xi(\tau)=\{w_1(\tau),\ldots,w_m(\tau)\}$. In order to calculate the trace we need to choose a basis in $\Xi(\tau)$ orthogonal in the sense of H_α. Let therefore ϕ be a basis element. We obtain for the corresponding diagonal entry of the matrix

$$\langle F'\phi,\phi\rangle_\alpha = \int_{-\infty}^0 e^{-\alpha x}\phi_{xx}\phi\,dx + V\int_{-\infty}^0 e^{-\alpha x}\phi_x\phi\,dx$$

$$-\phi'(0)\int_{-\infty}^0 e^{-\alpha x}U_x\phi\,dx = I_1+I_2+I_3. \tag{3.6}$$

We integrate I_1 by parts and obtain

$$I_1+I_2 = e^{-\alpha x}\phi_x\phi|_{-\infty}^0 + (\alpha+V)\int_{-\infty}^0 e^{-\alpha x}\phi_x\phi\,dx - \int_{-\infty}^0 e^{-\alpha x}\phi_x^2\,dx.$$

It can be easily seen via integration by parts that, since $\int_{-\infty}^0 e^{-\alpha x}\phi^2\,dx=1$,

$$\int_{-\infty}^0 e^{-\alpha x}\phi_x\phi\,dx = \frac{1}{2}e^{-\alpha x}\phi^2|_{-\infty}^0 + \frac{\alpha}{2}.$$

Therefore

$$\langle F'\phi,\phi\rangle_\alpha = e^{-\alpha x}\phi_x\phi|_{-\infty}^0 + (\alpha+V)\left(\frac{1}{2}e^{-\alpha x}\phi^2|_{-\infty}^0 + \frac{\alpha}{2}\right)$$

$$-\int_{-\infty}^0 e^{-\alpha x}\phi_x^2\,dx - \phi'(0)\int_{-\infty}^0 e^{-\alpha x}U_x\phi\,dx$$

$$= \phi_x\phi|_0 + (\alpha+V)\left(\frac{1}{2}\phi^2|_0 + \frac{\alpha}{2}\right)$$

$$-\int_{-\infty}^0 e^{-\alpha x}\phi_x^2\,dx - \phi'(0)\int_{-\infty}^0 e^{-\alpha x}U_x\phi\,dx. \tag{3.7}$$

In the choice of the basis elements we shall distinguish two possibilities, $\phi(0)=0$ (which defines an $m-1$-dimensional subspace), and otherwise. Since the trace is independent of the choice of orthonormal basis we can choose $m-1$ basis elements satisfying the above condition. In the case $\phi(0)=0$ we obtain

$$\langle F'\phi, \phi\rangle_\alpha = \frac{\alpha}{2}(\alpha+V) - \int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx. \tag{3.8}$$

Note that the terms with $\phi'(0)$ vanish since $\phi'(0)=\phi(0)/v=0$ in view of the boundary condition. For the sequel we shall select the parameter α sufficiently small in order to make (3.8) a negative value. Note that V is negative. We will next estimate the absolute value of (3.8). First we observe that for an arbitrary positive c

$$\begin{aligned} 2\int_{-\infty}^{0} e^{-\alpha x}\phi_x\phi\,dx &\leqslant c\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx + \frac{1}{c}\int_{-\infty}^{0} e^{-\alpha x}\phi^2\,dx \\ &= c\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx + \frac{1}{c}. \end{aligned}$$

Hence

$$\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx \geqslant \frac{2}{c}\int_{-\infty}^{0} e^{-\alpha x}\phi_x\phi\,dx - \frac{1}{c^2}. \tag{3.9}$$

From the identity

$$0 = e^{-\alpha x}\phi^2\big|_{-\infty}^{0} = \int_{-\infty}^{0} (e^{-\alpha x}\phi^2)'\,dx = 2\int_{-\infty}^{0} e^{-\alpha x}\phi_x\phi\,dx - \alpha,$$

we obtain $\int_{-\infty}^{0} e^{-\alpha x}\phi_x\phi\,dx = \alpha/2$. Thus, (3.9) yields

$$\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx \geqslant \frac{\alpha}{c} - \frac{1}{c^2}.$$

It is easy to see that for $c=2/\alpha$ the right hand side of last inequality achieves its maximum. Therefore $\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx \geqslant \alpha^2/4$. Finally for this type of the basis element we obtain

$$\langle F'\phi, \phi\rangle_\alpha = \frac{\alpha}{2}(\alpha+V) - \int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx \leqslant \frac{\alpha}{2}\left(\frac{\alpha}{2}+V\right).$$

If we choose $\alpha < 2v_0$ the value on the right will be negative. However, to secure that U_x belongs to H_α we need to take α smaller than $\frac{v_0}{2}$, since the elements of the attractor exhibit decay as $\exp(-\frac{v_0}{4}x)$, see (2.8).

Then

$$\langle F'\phi, \phi\rangle_\alpha \leqslant -\frac{3v_0^2}{16}. \tag{3.10}$$

We recall that v_0 is the lower bound on the absolute value of the interface velocity that is defined exclusively by the properties of the kinetics.

For the basis element with $\phi(0) \neq 0$ the corresponding trace component,

$$\begin{aligned}\langle F'\phi, \phi\rangle_\alpha &= e^{-\alpha x}\phi_x\phi|_0 + (\alpha+V)\left(\frac{1}{2}\phi^2|_0 + \frac{\alpha}{2}\right)\\ &\quad -\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx - \phi_x(0)\int_{-\infty}^{0} e^{-\alpha x}U_x\phi\,dx\\ &= (2/v+\alpha+V)\frac{1}{2}\phi^2(0) + (\alpha+V)\frac{\alpha}{2}\\ &\quad -\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx - \frac{\phi(0)}{v}\int_{-\infty}^{0} e^{-\alpha x}U_x\phi\,dx,\end{aligned} \tag{3.11}$$

is estimated from above as follows. First we estimate the last term:

$$\begin{aligned}\left|\frac{\phi(0)}{v}\int_{-\infty}^{0} e^{-\alpha x}U_x\phi\,dx\right| &\leqslant \frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx + \frac{a}{2}\frac{\phi^2(0)}{v^2}\int_{-\infty}^{0} e^{-\alpha x}\phi^2\,dx\\ &= \frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx + \frac{a}{2}\frac{\phi^2(0)}{v^2}\end{aligned} \tag{3.12}$$

where $a > 0$ will be chosen later.

Now we need the following interpolation inequality:

$$\begin{aligned}\phi^2(0) &= \int_{-\infty}^{0} (e^{-\alpha x}\phi^2)_x\,dx = 2\int_{-\infty}^{0} e^{-\alpha x}\phi\phi_x\,dx - \alpha\int_{-\infty}^{0} e^{-\alpha x}\phi^2\,dx\\ &\leqslant 2\int_{-\infty}^{0} e^{-\alpha x}\left[\frac{c\phi_x^2}{2} + \frac{\phi^2}{2c}\right]dx - \alpha\int_{-\infty}^{0} e^{-\alpha x}\phi^2\,dx\\ &= c\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx + 1/c - \alpha.\end{aligned} \tag{3.13}$$

Taking into account (3.12)–(3.13), the relation in (3.11) yields:

$$\begin{aligned}\langle F'\phi, \phi\rangle_\alpha &\leqslant \left(2/v+\alpha+V+\frac{a}{v^2}\right)\frac{1}{2}\,\phi^2(0)+(\alpha+V)\,\frac{\alpha}{2}\\ &\quad-\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx+\frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx\\ &\leqslant\left[\left(2/v+\alpha+V+\frac{a}{v^2}\right)\frac{1}{2}\,c-1\right]\int_{-\infty}^{0} e^{-\alpha x}\phi_x^2\,dx\\ &\quad+\left(2/v+\alpha+V+\frac{a}{v^2}\right)\frac{1}{2}\,(1/c-\alpha)\\ &\quad+(\alpha+V)\,\frac{\alpha}{2}+\frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx.\end{aligned}$$

In the last expression, given a, we select the constant $c>0$ to satisfy $c(2/v+\alpha+V+\frac{a}{v^2})/2\leqslant 1$, for instance $c\leqslant\frac{2\min v}{2+a/\min v}$ is sufficient. Then the chain of inequalities from above can be continued:

$$\begin{aligned}&\leqslant(1/c-\alpha)/c+(\alpha+V)\,\frac{\alpha}{2}+\frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx\\ &\leqslant(1/c-\alpha)/c+\frac{1}{2a}\int_{-\infty}^{0} e^{-\alpha x}U_x^2\,dx\leqslant(1/c-\alpha)/c+\frac{1}{2a}\,\|U_x\|_\alpha^2.\end{aligned}$$

We recall that $\|U_x\|_\alpha\leqslant CV_0/\sqrt{v_0}$. We choose $\alpha=v_0/3$, $c=\frac{2\min v}{2+a/\min v}$, and a that gives the minimum to the function $(1/c-v_0)/c+\frac{1}{2a}\frac{CV_0^2}{v_0}$ (in the sequel this minimum is denoted by μ). This results in the estimate

$$\langle F'\phi, \phi\rangle_\alpha\leqslant\mu. \tag{3.14}$$

The explicit form of μ is not important for our purposes but it can be shown to be on the order of V_0^2. Thus, employing the above estimates for the trace entries (3.10), (3.14) we can complete the estimate for the evolution of the volume element

$$Tr[F'(U(\tau),V(\tau))\circ Q_m(\tau)]=\sum_{i=1}^{m}\langle F'\phi_i, \phi_i\rangle_\alpha\leqslant\mu-3(m-1)\,v_0^2/16.$$

Taking

$$m>M=(16/3)\,\mu/v_0^2+1 \tag{3.15}$$

is sufficient for the trace to become negative. Note that $M\sim cV_0^2/v_0^2$.

Remark 2. All the computations of the present section have been performed in the absence of heat losses, $\gamma = 0$. If the heat loss is taken into account then the definition of F' from (3.2) has to be augmented by the term $-\gamma w$. It is easy to see that the version of (3.15) for the $\gamma > 0$ case yields a negative trace for a lower dimension,

$$m > M = \frac{16\mu/3}{v_0^2 + 16\gamma/3} + 1. \tag{3.16}$$

Note that $M \sim cV_0^2/(v_0^2 + 16\gamma/3)$.

In order to claim that the *nonlinear* evolution of the elementary m-dimensional volume, $m > M$, contracts the volume and therefore to obtain an upper bound for Hausdorff dimension, we need to demonstrate the differentiability of the semigroup solving the free-boundary problem with respect to the initial conditions.

4. DIFFERENTIABILITY OF THE SEMIGROUP

In this section we demonstrate that the nonlinear evolution of the volume is well approximated by its linear counterpart. This will allow us to utilize the trace estimate developed in the previous section. The desired result will be ensured by the differentiability with respect to the initial conditions of the semigroup solving the free-boundary problem.

In Section 2 we discussed the global existence result for the *classical* solutions of the free-boundary problem (1.1–1.3). However, the attractor is imbedded into a Hilbert space; in order to evaluate its dimension we need to extend existence theory to more general initial data, that belong to a Hilbert space.

The scheme of introduction of weak solutions is rather standard: we consider Cauchy sequences in the Hilbert norm of uniformly bounded continuous functions. We show that their images under the semigroup action form also a Cauchy sequence whose limit is designated as the weak solution. The proof is based on the following estimates:

***Proposition** 7. Let U and W be two orbits (i.e., two solutions of the problem (3.1) with initial data U_0, W_0): $U = T(t)\,U_0$, $W = T(t)\,W_0$, where U_0, W_0 are continuous, belong to H_α with* $\sup |U_0|$, $\sup |W_0| \leqslant R$. *Then for any* $t > 0$

$$\begin{gathered} \|U(t) - W(t)\|_\alpha \leqslant e^{Ct}\, \|U_0 - W_0\|_\alpha \\ \int_0^t \|U(t) - W(t)\|_{\alpha,1}^2 \, d\tau \leqslant e^{Ct}\, \|U_0 - W_0\|_\alpha^2 \end{gathered} \tag{4.1}$$

where C depends on R and α.

The above proposition allows us to pass to the limit and as a result we obtain weak solutions with H_α initial conditions that are $C([0, t], H_\alpha) \cap L_2([0, t], H^1_\alpha)$.

Proof. Let $U(x, t)$ and $W(x, t)$ be two solutions of the free boundary problem (in the frame attached to the free boundary)

$$U_t = U_{xx} - U_x(0, t)\, U_x, \qquad g^{-1}U(0, t) = -U_x(0, t), \qquad U(x, 0) = U_0(x),$$
$$W_t = W_{xx} - W_x(0, t)\, W_x, \qquad g^{-1}W(0, t) = -W_x(0, t), \qquad W(x, 0) = W_0(x).$$

The difference $w = U - W$ solves the following problem

$$w_t = w_{xx} - U_x(0, t)\, w_x - w_x(0, t)\, W_x, \tag{4.2}$$

$$w(x, 0) = U_0(x) - W_0(x), \tag{4.3}$$

$$w_x(0, t) = g^{-1}(W(0, t)) - g^{-1}(U(0, t)) = -(g^{-1}(\theta))'\, w(0, t). \tag{4.4}$$

We also observe that $-(g^{-1}(\theta))' < \text{const}$ while U, W, and their x-derivatives are uniformly bounded on the attractor.

We multiply the equation throughout by $e^{-\alpha x} w$ and integrate to obtain for the H_α norm $|\cdot|_\alpha$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2_\alpha &= \int_{-\infty}^0 e^{-\alpha x} w_{xx} w\, dx - U_x(0, t) \int_{-\infty}^0 e^{-\alpha x} w_x w\, dx \\ &\quad - w_x(0, t) \int_{-\infty}^0 e^{-\alpha x} W_x w\, dx \\ &= e^{-\alpha x} w_x w|_0 - |w_x|^2_\alpha + (\alpha - U_x(0, t)) \int_{-\infty}^0 e^{-\alpha x} w_x w\, dx \\ &\quad - w_x(0, t) \int_{-\infty}^0 e^{-\alpha w} W_x w\, dx.\end{aligned} \tag{4.5}$$

We need to estimate different terms in (4.5)

$$\begin{aligned}|w_x(0, t)\, w(0, t)| &\leqslant C w(0, t)^2 = C \int_{-\infty}^0 (e^{-\alpha x} w^2)_x\, dx \\ &= 2C \int_{-\infty}^0 e^{-\alpha x} w_x w\, dx - \alpha C \int_{-\infty}^0 e^{-\alpha x} w^2\, dx \\ &\leqslant C\left(\varepsilon_1 |w|^2_\alpha + \frac{1}{\varepsilon_1} |w_x|^2_\alpha - \alpha |w|^2_\alpha\right).\end{aligned}$$

Next,

$$\left|(\alpha - U_x(0,t))\int_{-\infty}^{0} e^{-\alpha x} w_x w\, dx\right| \leqslant (\alpha + |U_x(0,t)|)\left(\varepsilon_2\, |w|_\alpha^2 + \frac{1}{\varepsilon_2}\, |w_x|_\alpha^2\right)$$

$$\leqslant C_1\left(\varepsilon_2\, |w|_\alpha^2 + \frac{1}{\varepsilon_2}\, |w_x|_\alpha^2\right).$$

Also,

$$\left|w_x(0,t)\int_{-\infty}^{0} e^{-\alpha x} W_x w\, dx\right| \leqslant C_3\, |w(0,t)|\, |W_x|_\alpha\, |w|_\alpha$$

$$\leqslant C_4\left(\varepsilon_3\, |w|_\alpha^2 + \frac{1}{\varepsilon_3}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2\right)^{1/2} |w|_\alpha.$$

In the last inequality we set $\varepsilon_3 = \alpha$, to obtain

$$w_x(0,t)\int_{-\infty}^{0} e^{-\alpha x} W_x w\, dx \leqslant C_4\, \frac{1}{\sqrt{\alpha}}\, |w_x|_\alpha\, |w|_\alpha \leqslant C_5\left(\varepsilon_4\, |w|_\alpha^2 + \frac{1}{\varepsilon_4}\, |w_x|_\alpha^2\right).$$

Collecting the estimates for different terms we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\, |w|_\alpha^2 &\leqslant -|w_x|_\alpha^2 + C\left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2\right) \\ &\quad + C_1\left(\varepsilon_2\, |w|_\alpha^2 + \frac{1}{\varepsilon_2}\, |w_x|_\alpha^2\right) + C_5\left(\varepsilon_4\, |w|_\alpha^2 + \frac{1}{\varepsilon_4}\, |w_x|_\alpha^2\right) \\ &\leqslant -\frac{1}{2}\, |w_x|_\alpha^2 + C_0\, |w|_\alpha^2\end{aligned}$$

where on the last step we have chosen $C/\varepsilon_1 + C_1/\varepsilon_2 + C_5/\varepsilon_4 < 1/2$. We rewrite our last result as

$$\frac{d}{dt}\, |w|_\alpha^2 + |w_x|_\alpha^2 \leqslant C\, |w|_\alpha^2. \tag{4.6}$$

From this inequality we get first that $\frac{d}{dt}\, |w|_\alpha^2 \leqslant C\, |w|_\alpha^2$ yielding, by Gronwall's inequality, that

$$|w|_\alpha^2 \leqslant |w_0|_\alpha^2 \exp(Ct);$$

at the same time by rearranging and integrating (4.6) we obtain

$$\int_0^t |w_x|_\alpha^2 \, d\tau \leqslant \int_0^t \left(C\, |w|_\alpha^2 - \frac{d}{dt} |w|_\alpha^2 \right) d\tau \leqslant |w_0|_\alpha^2 \exp(Ct). \quad \square$$

Remark 3. If the initial data are in H_α^1 then a similar argument yields the following estimate analogous to (4.1):

$$\|U(t) - W(t)\|_{\alpha, 1} \leqslant e^{Ct} \, \|U_0 - W_0\|_{\alpha, 1}. \tag{4.7}$$

This guarantees well-posedness of the problem in H_α^1. The detailed arguments are somewhat technically involved and shall be presented elsewhere.

The free-boundary problem (1.1–1.3) is not well-posed in H_α, which is quite common for the semigroups generated by PDEs. However, differentiability of the semigroup can be demonstrated for initial data in $H_{\alpha, 1} \subset H_\alpha$. In $H_{\alpha, 1}$ the attractor can be viewed as a "functional-invariant set" in the sense of [Constantin *et al.* (1985)]. This is sufficient for the validity of the dimension estimate, see [Temam (1988), Chap. VII].

In fact we shall show that a result a little stronger than differentiability holds:

Theorem 8. *Let U and W be two orbits $U = T(t)\, U_0$, $W = T(t)\, W_0$, $U_0, W_0 \in H_{\alpha, 1}$. Then there exists $z(t)$ such that*

$$\|U(t) - W(t) - z(t)\|_\alpha \leqslant \text{const}\, \|U_0 - W_0\|_{\alpha, 1}^2$$

as $W_0 \to U_0$.

In this case the Frechét differential of $T(t)$ at the point U_0 is the mapping $z(0) = U_0 - W_0 \to z(t)$, where $z(t)$ solves the linearized problem.

Proof. The goal of the proof is to evaluate the difference between $w = U - W$ and its approximation by the differential. We define $z(x, t)$ as a solution of the free-interface problem linearized about the orbit $U(x, t)$:

$$\begin{gathered} z_t = z_{xx} - z_x(0, t)\, U_x - U_x(0, t)\, z_x, \\ z(0, t) = -g'(-U_x(0, t))\, z_x(0, t), \qquad z(x, 0) = U_0(x) - W_0(x), \end{gathered} \tag{4.8}$$

For the difference $y = w - z$ we have the following equations

$$\begin{gathered} y_t = y_{xx} - y_x(0, t)\, U_x - U_x(0, t)\, y_x, \\ y(0, t) = -g'(-U_x(0, t))\, y_x(0, t) + g''(\theta)\, w_x^2(0, t)/2, \qquad y(x, 0) = 0, \end{gathered} \tag{4.9}$$

We multiply the equation throughout by $e^{-\alpha x} y$ and integrate to obtain the following identity for the H_α norm $|\cdot|_\alpha$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|y|_\alpha^2 &= \int_{-\infty}^{0} e^{-\alpha x} y_{xx}\, y\, dx - U_x(0, t) \int_{-\infty}^{0} e^{-\alpha x} y_x\, y\, dx \\
&\quad - y_x(0, t) \int_{-\infty}^{0} e^{-\alpha x} U_x\, y\, dx \\
&= e^{-\alpha x} y_x\, y|_0 - |y_x|_\alpha^2 + (\alpha - U_x(0, t)) \int_{-\infty}^{0} e^{-\alpha x} y_x\, y\, dx \\
&\quad - y_x(0, t) \int_{-\infty}^{0} e^{-\alpha x} U_x\, y\, dx. \qquad (4.10)
\end{aligned}$$

We need to estimate different terms in (4.10)

$$\begin{aligned}
|y_x(0, t)\, y(0, t)| &\leqslant C y^2(0, t) + C w_x^2(0, t)\, |y(0, t)| \leqslant \frac{3}{2} C y^2(0, t) + \frac{1}{2} C w_x^4(0, t) \\
&= B_1\, y^2(0, t) + B_2 w_x^4(0, t) \\
&\leqslant B_1 \int_{-\infty}^{0} (e^{-\alpha x} y^2)_x\, dx + B_3 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^1 \right)^2 \\
&= 2B_1 \int_{-\infty}^{0} e^{-\alpha x} y_x\, y\, dx - \alpha B_1 \int_{-\infty}^{0} e^{-\alpha x} y^2\, dx \\
&\quad + B_3 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2 \right)^2 \\
&\leqslant B_1 \left(\varepsilon_1\, |y|_\alpha^2 + \frac{1}{\varepsilon_1}\, |y_x|_\alpha^2 - \alpha\, |y|_\alpha^2 \right) \\
&\quad + B_3 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2 \right)^2.
\end{aligned}$$

Next,

$$\begin{aligned}
\left| (\alpha - U_x(0, t)) \int_{-\infty}^{0} e^{-\alpha x} y_x\, y\, dx \right| &\leqslant |(\alpha - U_x(0, t))| \left(\varepsilon_2\, |y|_\alpha^2 + \frac{1}{\varepsilon_2}\, |y_x|_\alpha^2 \right) \\
&\leqslant C_1 \left(\varepsilon_2\, |y|_\alpha^2 + \frac{1}{\varepsilon_2}\, |y_x|_\alpha^2 \right).
\end{aligned}$$

Also,

$$\begin{aligned}\left| y_x(0,t) \int_{-\infty}^{0} e^{-\alpha x} U_x\, y\, dx \right| &\leqslant (C_3\, |y(0,t)| + B_4 w_x^2(0,t)) \int_{-\infty}^{0} e^{-\alpha x}\, |U_x\, y|\, dx \\ &\leqslant (C_3\, |y(0,t)| + B_4 w_x^2(0,t))\, |y|_\alpha\, |U_x|_\alpha \\ &\leqslant C_4 \left(\varepsilon_3\, |y|_\alpha^2 + \frac{1}{\varepsilon_3}\, |y_x|_\alpha^2 - \alpha\, |y|_\alpha^2 \right)^{1/2} |y|_\alpha \\ &\quad + B_5 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2 \right) |y|_\alpha\end{aligned}$$

where the constants C_4 and B_5 include the factor $|U_x|_\alpha$. In the last inequality we set $\varepsilon_3 = \alpha$, $\varepsilon_1 = \alpha$, to obtain

$$\begin{aligned}y_x(0,t) \int_{-\infty}^{0} e^{-\alpha x} U_x w\, dx &\leqslant C_4\, \frac{1}{\sqrt{\alpha}}\, |y_x|_\alpha\, |y|_\alpha + B_5\, \frac{1}{\alpha}\, |w_x|_\alpha^2\, |y|_\alpha \\ &\leqslant C_5 \left(\varepsilon_4\, |y|_\alpha^2 + \frac{1}{\varepsilon_4}\, |y_x|_\alpha^2 \right) + B_6 (|w_x|_\alpha^4 + |y|_\alpha^2).\end{aligned}$$

Collecting the estimates for different terms we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\, |y|_\alpha^2 &\leqslant -|y_x|_\alpha^2 + C \left(\varepsilon_1\, |y|_\alpha^2 + \frac{1}{\varepsilon_1}\, |y_x|_\alpha^2 - \alpha\, |y|_\alpha^1 \right) \\ &\quad + B_3 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2 \right)^2 + C_1 \left(\varepsilon_2\, |y|_\alpha^2 + \frac{1}{\varepsilon_2}\, |y_x|_\alpha^2 \right) \\ &\quad + C_5 \left(\varepsilon_4\, |y|_\alpha^2 + \frac{1}{\varepsilon_4}\, |y_x|_\alpha^2 \right) + B_6 (|w_x|_\alpha^4 + |y|_\alpha^2) \\ &\leqslant -\frac{1}{2}\, |y_x|_\alpha^2 + C_6\, |y|_\alpha^2 + B_3 \left(\varepsilon_1\, |w|_\alpha^2 + \frac{1}{\varepsilon_1}\, |w_x|_\alpha^2 - \alpha\, |w|_\alpha^2 \right)^2 \\ &\quad + B_6\, |w_x|_\alpha^4.\end{aligned}$$

We rewrite our last result as

$$\frac{d}{dt}\, |y|_\alpha^2 + |y_x|_\alpha^2 \leqslant C\, |y|_\alpha^2 + B_6\, |w_x|_\alpha^4 \tag{4.11}$$

from where it is clear that

$$\frac{d}{dt}\,|y|_{\alpha}^{2} \leqslant C\,|y|_{\alpha}^{2} + B_6\,|w_x|_{\alpha}^{4}. \tag{4.12}$$

By Gronwall's inequality it yields

$$\begin{aligned}|y|_{\alpha}^{2} &\leqslant B_6 \exp(Ct) \int_0^t |w_x|_{\alpha}^{4} \exp(-C\tau)\, d\tau \\ &\leqslant B_6 \exp(Ct) \int_0^t |w|_{\alpha,1}^{4} \exp(-C\tau)\, d\tau \\ &\leqslant B_7 \exp(Ct)\, |w_0|_{\alpha,1}^{4}.\end{aligned}$$

In the above estimate we utilized (4.7). □

Finally, the estimate for the dimension of the linear volume element and differentiability of the semigroup yield the estimate for the Hausdorff dimension of the attractor:

Theorem 9. *The Hausdorff dimension of the attractor (functional invariant set) $\mathscr{A}$ is no larger than M estimated above (3.16).*

In conclusion it is worth mentioning that the estimate exhibits a transparent and physically natural dependence of the dimension $M \sim cV_0^2/(v_0^2 + c\gamma)$ on the heat loss and characteristics of the kinetics which are the defining factors of the dynamics.

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REFERENCES

Brauner, C.-M., Hulshof, J., and Lunardi, A. (2000). A general approach to stability in free boundary problems. *J. Differential Equations* **164**, 16–48.

Brauner, C.-M., and Lunardi, A. (2000). Instabilities in a two-dimensional combustion model with free boundary. *Arch. Ration. Mech. Anal.* **154**, 157–182.

Chen, X., and Reitich, F. (1992). Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling. *J. Math. Anal. Appl.* **164**, 350–362.

Constantin, P., Foias, C., and Temam, R. (1985). Attractors representing turbulent flows. *Memoirs of AMS* **53**, no. 314.

Din, S., Tao, Y., and Yin, H.-M. (1997). A chemical diffusion process with reaction taking place at free boundary. *Canad. Appl. Math. Quaterly* **5**, 49–74.

Frankel, M., Roytburd, V., and Sivashinsky, G. (1994). A sequence of period doubling and chaotic pulsations in a free boundary problem modeling thermal instabilities. *SIAM J. Appl. Math.* **54**, 1101–1112.

Frankel, M., and Roytburd, V. (1994). Dynamical portrait of a model of thermal instability: Cascades, chaos, reversed cascades and infinite period bifurcations. *Int. J. Bifurcation Chaos* **4**, 579–593.

Frankel, M., and Roytburd, V. (1995). Finite-dimensional model of thermal instability. *Appl. Math. Lett.* **8**, 39–44.

Frankel, M., Kovačič, G., Roytburd, V., and Timofeyev, I. (2000). Finite-dimensional dynamical system modeling thermal instabilities. *Physica D* **137**, 295–315.

Frankel, M., and Roytburd, V. (2002). Finite-dimensional attractors for a free-boundary problem with a kinetic condition. *Appl. Math. Lett.* **15**, 83–87.

Frankel, M., and Roytburd, V. (2002). Compact attractors for a Stefan problem with kinetics. *EJDE* **21**, 1–27.

Friedman, A. (1964). *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Englewood Cliffs, NJ.

Gol'berg, S. M., and Tribelskii, M. I. (1985). On laser induced evaporation of nonlinear absorbing media. *Zh. Tekh. Fiz. (Sov. Phys.-J. Tech. Phys.)* **55**, 848–857.

Langer, J. S. (1987). Lectures in the theory of pattern formation. In Souletie, J., Vannimenus, J., and Stora, R. (eds.), *Chance and Matter*, Elsevier Science Publishers.

Luckhaus, S. (1990). Solutions for the two-phase Stefan problem with the Gibbs–Thomson law for the melting temperature. *Euro. J. of Appl. Math.* **1**, 101–111.

Matkowsky, B. J., and Sivashinsky, G. I. (1978). Propagation of a pulsating reaction front in solid fuel combustion. *SIAM J. Appl. Math.* **35**, 230–255.

Munir, Z. A., and Anselmi-Tamburini, U. (1989). Self-propagating exothermic reactions: The synthesis of high-temperature materials by combustion. *Mater. Sci. Rep.* **3**, 277–365.

Radkevich, E. V. (1991). The Gibbs–Thomson correction and conditions for the existence of a classical solution of the modified Stefan problem. *Soviet Math. Dokl.* **41**, 274–278.

Temam, R. (1988). Infinite-dimensional dynamical systems. In *Mechanics and Physics*, Springer Verlag, New York.

Van Saarloos, W., and Weeks, J. (1984). Surface undulations in explosive crystallization: A nonlinear analysis of a thermal instability. *Physica D* **12**, 279–294.

Yin, H.-M. (1993). Blowup and global existence for a non-equilibrium phase change process. In Friedman, A., and Spruck, J. (eds.), *Variational and Free Boundary Problems*, The IMA Volumes in Mathematics and its Applications, Vol. 53, Springer-Verlag, New York, 195–204.