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Finite-Dimensional Attractors for a Free Boundary Problem with a Kinetic Condition

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Abstract—For a one-phase free boundary problem with kinetics, a proof of existence of a compact attractor of finite Hausdorff dimension is outlined.

Keywords—Free boundary problems, Exothermic phase transitions, Hausdorff dimension.

1. INTRODUCTION

In this paper, we announce our recent results on asymptotic behavior of solutions of the modified one-phase Stefan problem in one spatial dimension:

$$u_t = u_{xx} - \gamma u, \qquad -\infty < x < s(t), \tag{1}$$

$$\left(\frac{\partial u}{\partial x}\right)\bigg|_{x=s(t)} = -V(t), \qquad g\left(u|_{x=s(t)}\right) = V(t), \tag{2}$$

$$u(x,0) = u^0(x). \tag{3}$$

Here $u(x,t)$ represents the temperature, and the damping term $\gamma \geq 0$ is the volumetric heat loss; the two boundary conditions overdetermine the problem and allow us to find the free boundary whose position is $s(t)$ and velocity, $V(t) = \dot{s}(t)$. This problem arises naturally as a mathematical model of phenomena that play an important role in many phase transition type processes (such as certain types of combustion [1], solidification with undercooling [2], laser induced evaporation [3], etc.). For these exothermic processes which are characterized by production of heat at the interface, the basic mechanism that controls dynamics of the interface is determined by the

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balance between the heat release due to the kinetics and its dissipation by the medium. The first boundary condition in (2) represents the energy balance at the free boundary: the heat released as the boundary moves should be transferred into the medium. The second, kinetic, boundary condition is a manifestation of the nonequilibrium nature of the problem; its analog for the classical Stefan problem is just $u|_{x=s(t)} = 0$. In the context of condensed phase combustion, for example, the kinetic boundary condition expresses the dependence of the propagation velocity on the flame temperature.

Our principal result is a proof of existence of a compact attractor of finite Hausdorff dimension for the dynamics governed by (1)–(3). This work was motivated by the available results of DNS and, in particular, by our prior "experimental" work in [4,5] (see also [6]). On one hand, in [4], we presented a series of DNS for different versions of the kinetic function that demonstrated a very fast convergence to a whole range of dynamical patterns such as a Hopf bifurcation, period doubling cascades, a Shilnikov bifurcation, etc. Most of these patterns are well known and have been observed for the finite-dimensional dynamical systems (ODEs). On the other hand, in [5], we designed a 3×3 system of ODEs which is a pseudo-spectral approximation to the free boundary problem; the dynamics of the latter mimic those of the former to an amazing degree. These experimental observations led to the conjecture that the asymptotic dynamics of (1)–(3) are finite dimensional.

2. EXISTENCE OF CLASSICAL SOLUTIONS

A short-time solution of the free boundary problem will be sought in the form of a superposition of heat potentials,

$$u(x,t) = \int_0^t \tilde{G}(x, s(\tau), t-\tau)\varphi(\tau)\,d\tau + \int_{-\infty}^0 \tilde{G}(x,\xi,t)u^0(\xi)\,d\xi, \tag{4}$$

where $\tilde{G} = e^{-\gamma(t-\tau)}G$ is the fundamental solution of the heat equation with damping, G is the Gaussian kernel

$$G(x,\xi,t-\tau) \equiv G(x,t,\xi,\tau) = \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\}[4\pi(t-\tau)]^{-1/2}.$$

The density of the single layer potential φ and the front position $s(t)$ are to be determined. It is not hard to see that φ must have a $1/\sqrt{t}$ singularity at 0. In spite of this singularity, it can be proved that the single-layer potential possesses the standard jump property (which, of course, is well known for φ continuous). The jump condition, together with the kinetic boundary condition, result in the following system of equations:

$$u(s(t),t) = g^{-1}(V(t)) = \int_0^t \tilde{G}(s(t), s(\tau), t-\tau)\varphi(\tau)\,d\tau + \int_{-\infty}^0 \tilde{G}(s(t),\xi,t)u^0(\xi)\,d\xi, \tag{5}$$

$$u_x(s(t),t) = -V(t) = \frac{\varphi}{2} - \int_0^t \tilde{G}_\xi(s(t), s(\tau), t-\tau)\varphi(\tau)\,d\tau - \int_{-\infty}^0 \tilde{G}_\xi(s(t),\xi,t)u^0(\xi)\,d\xi. \tag{6}$$

The equations should be supplemented by the compatibility initial conditions: $V(0) = g(u^0(0))$, $\lim_{t\to 0}\sqrt{t}\varphi(t) = u^0(0)/\sqrt{\pi}$.

THEOREM 1. *Let $g < 0$ be continuously differentiable monotone decreasing, $u^0 \in C(-\infty, 0]$, $u^0 > 0$. Then the problem in (5),(6) has a unique solution $\{V, \varphi\}$ such that V and $\sqrt{t}\varphi(t)$ are continuous on $[0,\sigma]$ for some $\sigma > 0$, where σ depends only on $\sup u^0$. A solution to the free boundary problem is determined by V, φ via representation (4).*

The theorem differs from our earlier result as well as from other existence results [7–9] in not requiring any smoothness from the initial data. This result is crucial if one is to establish compactifying properties of the evolution. The proof is based on a contraction argument for σ small enough. It should be noted however that the singularity in the potential density precludes a simple-minded iteration scheme from being a contraction. Roughly speaking, the contraction rate for nonsingular densities is on the order of $\sqrt{\sigma}$. The $1/\sqrt{t}$ singularity leads to a "cancellation" and prevents us from making the rate of contraction small.

To overcome this difficulty, we introduce a two-step iteration scheme, analogous to the coordinate descent. Namely, given $\{\varphi, V\}$, we use (6) to find the update $\tilde{\varphi}$ for φ, then given $\{\tilde{\varphi}, V\}$, equation (5) is used to update V. The convergence is in the norm $\max\{\|\varphi(.)\sqrt{\cdot}\|_{C[0,\sigma]}, \|V(.)\|_{C[0,\sigma]}\}$.

To prove global existence, we need to impose an extra condition on the kinetic function

$$-V_0 \leq g \leq -v_0 < 0. \tag{7}$$

The lower bound is satisfied for the standard Arrhenius kinetics (actually the result holds even if g has a sublinear growth), while the upper bound v_0 corresponds to the "ignition velocity": the model is valid only for moving fronts. Global existence is guaranteed by the following *a priori* estimate (cf. [10]).

LEMMA 2. *Let u be a solution of the free boundary problem, then the interface temperature $U(\tau) = u(s(\tau), \tau)$ is uniformly bounded $|U(\tau)| \leq R_{fb} + 2e^{-\gamma t}\|u^0\|_{C(-\infty,0)}$, where the constant $R_{fb} = g^{-1}(-V_0/2)V_0/(v_0 + \sqrt{\gamma})$ is totally determined by the kinetic function. By the maximum principle, u is also uniformly bounded.*

3. A COMPACT ATTRACTOR AND ITS HAUSDORFF DIMENSION

The estimate in Lemma 2 and other useful estimates are based on the following representation for the solution:

$$\begin{aligned} u(x,t) = &\int_0^t \tilde{G}(x, s(\tau), t-\tau)\,[-V(\tau) + U(\tau)V(\tau)]\,d\tau \\ &- \int_0^t \frac{\partial \tilde{G}}{\partial \xi}(x, s(\tau), t-\tau)U(\tau)\,d\tau + \int_{-\infty}^0 \tilde{G}(x,\xi,t)u^0(\xi)\,d\xi, \end{aligned} \tag{8}$$

which is obtained by integrating Green's identity over the domain $\xi < s(\tau)$, $0 < \tau < t$. Since both U and V are determined by the initial conditions, the representation can be thought of as the time evolution of the initial temperature distribution u^0: $u(t) = T(t)u^0$. We understand the evolution as taking place for the functions on the fixed interval $(-\infty, 0)$. This is equivalent to the introduction of the moving coordinate system attached to the free boundary $x' = x - s(t)$. We split the semigroup operator T into two parts: the contribution of the free boundary and that of the initial data

$$T_1(t)u^0 = \int_0^t \tilde{G}(x, s(\tau), t-\tau)\,[-V(\tau) + U(\tau)V(\tau)]\,d\tau - \int_0^t \frac{\partial \tilde{G}}{\partial \xi}(x, s(\tau), t-\tau)U(\tau)\,d\tau, \tag{9}$$

$$T_2(t)u^0 = e^{-\gamma t}\int_{-\infty}^0 G\left(x', \xi - s(t), t\right) u^0(\xi)\,d\xi. \tag{10}$$

As a basic metric space, we choose a ball in the space $C(-\infty, 0]$:

$$X = \{u \in C(-\infty, 0];\ \|u\| = \sup|u(x')| \leq N\},$$

where the radius N is large enough (it suffices to take $N > R_{fb} + 2R_{\text{abs}}$ where R_{abs} is the radius of the absorbing ball which is estimated in the following proposition. Note that by Lemma 2, the evolution of any ball B_R of radius $R \leq (N - R_{fb})/2$ stays in X for all time.

The following result establishes existence of an absorbing set for the evolution.

PROPOSITION 3.

(i) *The semigroup T_2 is uniformly contracting:*

$$r_X(t) = \sup_{u^0 \in X} \left\|T_2(t)u^0\right\| \to 0, \qquad \text{as } t \to \infty.$$

(ii) *There exists a constant R_{abs}, totally determined by the kinetics such that any ball $B_a = \{u \in X;\ \|u\| \leq a\}$, where $a = R_{\text{abs}} + \varepsilon < N$ is an absorbing set for the ball B_R with respect to the evolution by T_1 (and T).*

The first assertion of the proposition is obvious, while the second is obtained by substituting the estimate of Lemma 2 into the integral representation (9).

Next we note that the boundary contribution to the evolution, i.e., the family of operators $T_1(t)$, is *uniformly compact*. Namely, the following proposition holds.

PROPOSITION 4. *There exists $t_0 > 0$ such that $\bigcup_{t \geq t_0} T_1(t)X$ is relatively compact in X.*

The proof of the proposition contains the two basic ingredients. We establish certain estimates on the functions $T_1(t)u$, and their first spatial derivatives, uniformly in $u \in X$, that are valid for any $t \geq t_0 > 0$. Next we demonstrate that the set determined by the estimates is relatively compact.

We note that for sufficiently small $t_0 > 0$ and any $u^0 \in X$, $|(T(t)u^0)_x| \leq C$ for $t \geq t_0$, $x \in (-\infty, 0]$. The estimate for a fixed small t_0 is obtained through differentiation of the single-layer representation (4). It is extended to all t since u_x is a solution of the Dirichlet problem for the heat equation in the domain $t \geq t_0$, $x \in (-\infty, 0]$, with initial data $u_x(\cdot, t_0)$ and boundary values V, and therefore, is uniformly bounded inside the domain by the maximum principle. From here, it is easy to deduce a similar estimate for T_1: $|(T_1(t)u^0)_x| \leq C$ uniformly for all $u^0 \in X$. Therefore, the set $\bigcup_{t \geq t_0} T_1(t)X$ is equicontinuous. From representation (9), it is not hard to see that

$$\left|\left(T_1(t)u^0\right)(x')\right| \leq e^{-v_0|x'|/4}\left(R_{\text{abs}} + CNe^{-\gamma t}\right),$$

which yields the uniform boundedness and uniform decay as $x \to -\infty$. Now it is easy to prove a version of the Arzela-Ascoli theorem appropriate for $(-\infty, 0]$ that completes the proof.

The properties of the evolution operator $T(t)$ described in the above propositions allow us to apply the abstract general result (see, for example, [11, Chapter 1]) that in our situation can be stated as follows.

THEOREM 5. *The continuous semigroup $T(t)$, $T(t) = T_1(t) + T_2(t)$ with $T_1(t)$ uniformly compact and $T_2(t)$ uniformly contracting has the following properties: the ω-limit set A of the absorbing set B_a is a compact attractor for the metric space X; A is the maximal attractor in X, and it is connected.*

In conclusion, we note that we also proved that the attractor has a finite Hausdorff dimension. In its outline, the argument follows closely the ideas of [12] (see also [11]). We imbed the attractor in the Hilbert space $H_\alpha = \{f \mid e^{-\alpha x} f \in L_2(-\infty, 0)\}$, $\alpha < v_0/4$, and study evolution of infinitesimal volume elements along the trajectories in the attractor. This reduces to the evaluation of the trace of finite-dimensional projections of the linearization. The space limitations of this letter preclude us from presenting the (numerous) technical details; we just state the result.

THEOREM 6. *Let the kinetic function satisfy $-\frac{dg^{-1}}{dV} \geq \underline{\nu} > 0$. Then the compact attractor A has the Hausdorff dimension $m > M = cV_0^2/v_0^2$, where c can be estimated explicitly.*

These results will be published in complete detail elsewhere.

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