

A stretch-temperature model for flame-flow interaction

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Received 10 May 2006; received in revised form 1 September 2006; accepted 1 September 2006

Available online 28 September 2006

Communicated by C.R. Doering

Abstract

The classical relation between the flame speed and the stretch, employed in modeling flame-flow interaction, is valid only for positive Markstein lengths (high Lewis numbers). At negative Markstein lengths (low Lewis numbers) the corresponding dynamical system suffers short-wavelength instability, making the associated initial value problem ill-posed. In this study the difficulty is resolved by incorporation of higher-order effects using a geometrically-invariant extrapolation from the linear analysis data. As a result one ends up with a reduced model based on a coupled system of second-order dynamic equations for the flame interface and its temperature. As an illustration the new model is applied for description of diffusively unstable stagnation-point flow flames.

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PACS: 47.20.Ky; 47.20.Ma; 82.40.Py

Keywords: Flame instability; Flame stretch; Surface dynamics

1. Introduction

When the hydrodynamic length is much larger than the flame thickness, the premixed flame may be regarded as an interface, convected by the unsteady flow field while advancing towards the unburned gas at a normal velocity v_F . In the constant density limit when the flow-field $\mathbf{u}(\mathbf{x}, t)$ is prescribed (passive propagation) the flame dynamics is governed by the equation,

$$v_n - \mathbf{u} \cdot \mathbf{N} = v_F \quad (\nabla \cdot \mathbf{u} = 0). \quad (1)$$

Here v_n is the normal velocity of the interface relative to the laboratory frame of reference, \mathbf{N} is the unit normal to the interface directed towards the unburned gas, and v_F is the normal velocity of the interface relative to a local fluid element; v_F de-

pends on the local curvature of the interface as well as the local strain field.

For weakly curved flames evolving through large-scale flow fields [1,2],

$$v_F = v_L(1 - l_M \mathcal{K}), \quad (2)$$

where v_L represents the laminar flame velocity in the absence of strain and curvature effects. \mathcal{K} is the stretch, the combined effect of curvature and strain.

For simplicity of the mathematical expressions involved, yet without much detriment to general understanding, the further discussion is restricted to the planar geometry only. The relation specifying the stretch may then be written as,

$$v_L \mathcal{K} = -v_L \kappa + \mathbf{u}_l \cdot \mathbf{T}. \quad (3)$$

Here the subscript l labels the arc-length derivative, \mathbf{T} is the unit tangential vector, $\kappa = -\nabla \cdot \mathbf{N}$ is the curvature of the interface. l_M is the Markstein length whose value depends on physico-chemical parameters of the system. In the constant

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density, high activation energy, near-equidiffusive approximation [3],

$$l_M = l_{th} \left(1 + \frac{1}{2} \beta (Le - 1) \right), \quad (4)$$

where l_{th} is the planar flame thickness; β , Le are the Zeldovich and Lewis numbers, respectively.

Eqs. (1), (2) define a second-order equation. For positive l_M (high Lewis numbers) the short-wavelength disturbances decay exponentially ensuring the well-posedness of the associated initial-value problem. However, if l_M is negative (small Lewis numbers) the short-wavelength disturbances undergo exponential growth, making the initial-value problem ill-posed, and the whole model incapable of securing a sensible dynamical picture.

The short-wavelength instability is a trace of the well-known flame cellularity occurring at low Lewis numbers and first explained by Zeldovich and co-workers [4]. Specifically, diffusion of the deficient reactant tends to amplify flame corrugations by enhancing (reducing) the reaction at the flame front convex (concave) towards the fresh mixture. The heat transfer exerts the opposite, that is, smoothing influence. At relatively high molecular diffusivity (low Lewis numbers) its destabilizing impact becomes a dominating factor. As a result the flame assumes the cellular structure.

An adequate description of cellular flames requires incorporation of higher-order gradient effects absent in the first-order model based on Eqs. (1), (2). Such higher-order models have long been developed for the flames evolving in quiescent mixtures ($\mathbf{u} = 0$). Here, depending on the asymptotic strategy adopted, one ends up either with a single fourth-order equation for the flame-interface [5,6] or with a system of second-order equations for the flame interface and its temperature [7,8]. The objective of the present study is an extension of the higher-order models to incorporate effects due to the background flow-field, which, to our knowledge, has not previously been undertaken.

2. Formulation

As a starting point we adopt the conventional constant density, high activation energy, near equidiffusive formulation where the reaction rate is modeled by the surface δ -function, δ_S . In suitably chosen units, the corresponding system of equations reads [3],

$$\theta_\tau + \mathbf{U} \cdot \nabla \theta = \nabla^2 \theta + \exp(\psi) \delta_S, \quad (5)$$

$$\psi_\tau + \mathbf{U} \cdot \nabla \psi = \nabla^2 (\psi - \alpha \theta). \quad (6)$$

Far ahead of the flame interface S , the quantities θ , $\psi \rightarrow 0$, whereas far behind ψ grows no faster than polynomially. Moreover, $\theta \equiv 1$ in the wake of the flame front.

$$\theta = \lim_{\substack{\beta \rightarrow \infty \\ Le \rightarrow 1}} ((T - T_u)/(T_b - T_u)),$$

$$\psi = \lim_{\substack{\beta \rightarrow \infty \\ Le \rightarrow 1}} \left(\frac{1}{2} \beta \Delta \mathcal{H} / c_p (T_b - T_u) \right) \quad (7)$$

are the reduced non-dimensional temperature and the excess enthalpy evaluated in the limit $\beta \rightarrow \infty$, $Le \rightarrow 1$, while the product $\alpha = \frac{1}{2} \beta (1 - Le)$ remains finite. T , T_u , T_b are the local, initial and final temperatures, respectively. $\Delta \mathcal{H} = c_p (T_b - T_u) [(T - T_b)/(T_b - T_u) + C/C_u]$ is the excess enthalpy; C , C_u are the local and the initial concentrations of the deficient reactant; c_p is the specific heat, and $\mathbf{U} = \mathbf{u}/v_L$, scaled flow velocity. The scaled spatio-temporal coordinates are referred to l_{th} and l_{th}/v_L , respectively.

For further discussion it is convenient to introduce a new quantity: $\Theta = \psi|_S$. Since at the flame interface the deficient reactant is fully consumed ($C = 0$),

$$\Theta = \lim_{\substack{\beta \rightarrow \infty \\ Le \rightarrow 1}} \left(\frac{1}{2} \beta (T - T_b)/(T_b - T_u) \right). \quad (8)$$

The latter therefore may be regarded as a scaled excess temperature of the interface.

The model (5), (6) while apparently somewhat unusual is in fact readily derived from the conventional reaction–diffusion–advection system comprised of two parabolic equations for the gas temperature (T) and the deficient reactant concentration (C), and coupled through the Arrhenius reaction rate term. Details can be found in Ref. [3].

3. Higher-order model

An extension of Eq. (2) accounting for higher-order effects may be either derived using an appropriate gradient expansion in the intrinsic coordinates, or may be constructed heuristically as a geometrically invariant extrapolation of the results obtained for freely evolving flames. In this Letter we adopt the latter strategy as less formalistic and more appealing physically. The advantage of a suitably arranged heuristic approach is that it straightforwardly exposes the interplay between the different mechanisms involved. In the perturbative derivation the origin of the individual ingredients constituting the final outcome is often difficult to trace due to the heavy algebra involved (see also Section 5).

We begin with the case of the quiescent mixture ($\mathbf{U} = 0$) where the system (5), (6) allows for a planar traveling wave (in the direction, say, \mathbf{N} with unit velocity) solution:

$$\theta^0 = \begin{cases} \exp(-y), & y > 0, \\ 1, & y < 0, \end{cases} \quad \psi^0 = \begin{cases} -\alpha y \exp(-y), & y > 0, \\ 0, & y < 0, \end{cases} \quad \Theta^0 = 0, \quad (9)$$

where y is the (signed) distance from the moving interface. The conventional linear stability analysis of the traveling wave solution yields the following set of relations between the perturbations of the interface δF , and its temperature $\delta \Theta$, valid in the long-wavelength limit,

$$(\delta F)_\tau = (\delta F)_{ss} + \delta \Theta, \quad (10)$$

$$(\delta \Theta)_\tau = (\delta \Theta)_{ss} - \frac{\alpha}{4} (\delta F)_{ss} - \frac{1}{4} \delta \Theta. \quad (11)$$

One can easily check that the instability sets in at $\alpha > \alpha_{cr} = 1$.

Note that for infinitesimally small perturbations of the traveling wave the coordinate s can be interpreted as either the usual Cartesian coordinate along the flat interface or the arc-length of the curved interface. Based on this observation and the coordinate invariance of the original free-interface problem (5), (6) the system (10), (11) suggests the following coordinate-free model for the flame interface dynamics,

$$V_n = 1 + \Theta + \kappa, \quad (12)$$

$$\mathcal{D}_\tau \Theta = \Theta_{ss} - \frac{\alpha}{4} \kappa - \frac{1}{4} \Theta, \quad (13)$$

where $V_n = v_n/v_L$, $\kappa = \kappa/l_{th}$ are the scaled normal velocity and curvature, respectively. \mathcal{D}_τ is the temporal derivative along the normal \mathbf{N} ; there is a simple connection between the normal time-derivative and the conventional partial time-derivative at constant s [9],

$$\mathcal{D}_\tau \Theta = \Theta_\tau + (\mathcal{D}_\tau s) \Theta_s, \quad (14)$$

where

$$\mathcal{D}_\tau s = - \int_0^s \kappa(\sigma, \tau) V_n(\sigma, \tau) d\sigma. \quad (15)$$

In the presence of the background flow \mathbf{U} the normal velocity in Eq. (12) should naturally be replaced by the flame speed relative to the fluid element $V_n - \mathbf{U} \cdot \mathbf{N}$, and $\mathcal{D}_\tau \Theta$ augmented by $(\mathbf{U} \cdot \mathbf{T}) \Theta_s$ reflecting advection of the interfacial temperature by the tangential flow field. The curvature term $-\kappa$ is a part of the total stretch (3), and, therefore, should be replaced by

$$K = l_{th} \mathcal{K} = -\kappa + \mathbf{U}_s \cdot \mathbf{T} = -(1 + \mathbf{U} \cdot \mathbf{N}) \kappa + (\mathbf{U} \cdot \mathbf{T})_s. \quad (16)$$

As a result one ends up with the following flame-flow model valid for general Lewis numbers,

$$V_n = \mathbf{U} \cdot \mathbf{N} + 1 + \Theta - K, \quad (17)$$

$$\mathcal{D}_\tau \Theta + (\mathbf{U} \cdot \mathbf{T}) \Theta_s = \Theta_{ss} + \frac{\alpha}{4} K - \frac{1}{4} \Theta. \quad (18)$$

The right of Eq. (2) therefore acquires a new term proportional to the interfacial excess temperature, governed by its own equation.

4. Cellular flames in the stagnation-point flow

As an illustration, in this section, the new $K-\Theta$ model is employed to describe the classical system of the diffusively unstable flame ($\alpha > 1$) held in the stagnation-point flow $\mathbf{U}(x, y) = (q(x - L/2), -qy)$ [10–12] (Fig. 1). Here q is the appropriately scaled flow intensity. The system is considered over half-strip ($0 < x < L$, $y < 0$) with the zero-gradient boundary conditions for the flame interface $y = F(x, \tau)$ and its excess temperature $\Theta(x, \tau)$,

$$F_x(0, \tau) = \Theta_x(0, \tau) = 0, \quad F_x(L, \tau) = \Theta_x(L, \tau) = 0. \quad (19)$$

The problem is solved numerically for $\alpha = 2.6$, $L = 800$, $0.01 \leq q \leq 0.02$. It is found, as expected [12], that for sufficiently high flow intensity ($q = 0.02$) the flame interface is

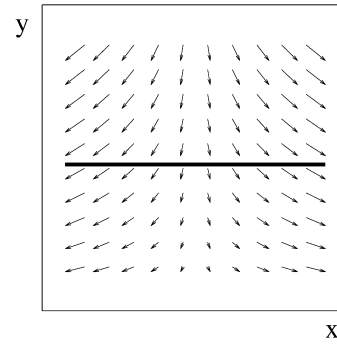


Fig. 1. Diagram of premixed planar flame (bold line) stabilized in stagnation-point flow field. The upstream flow corresponds to the fresh mixture, the downstream to the burned gas.

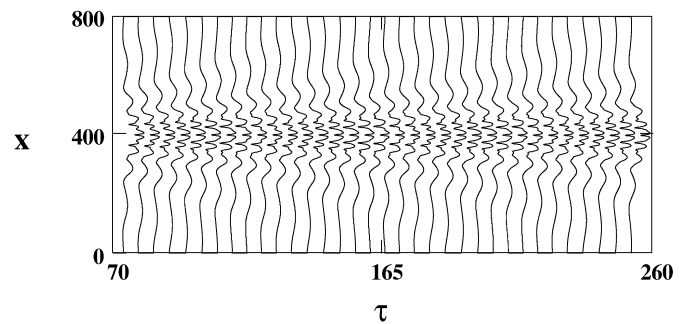


Fig. 2. Temporal evolution of the cellular flame.

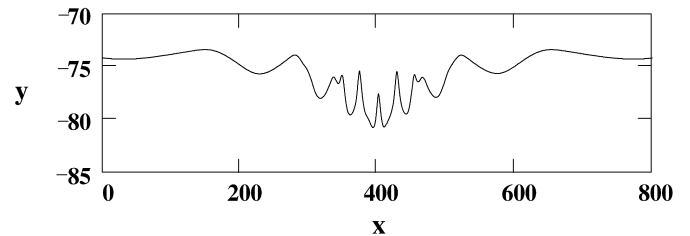


Fig. 3. An instantaneous profile of the flame interface ($\tau = 150$).

planar and located near $y = -1/q$. However, if q is small enough ($q = 0.014$) the central part of the flame becomes cellular with the cells in a state of permanent irregular self-motion similar to that occurring in freely propagating cellular flames. The concentration of the cells near the central stream-line is a new effect, which appears to be beyond the scope of the weakly nonlinear model explored previously [12]. The localization may be explained as follows. The velocity gradient along the flame interface stretches the flame corrugations away from the center-line thereby reducing the associated wave-number. According to the linear analysis the low wave-number corrugations are characterized by a lower growth rate. This results in a lower amplitude of the cells located farther from the center-line.

Fig. 2 shows the temporal evolution of the cellular flame at $q = 0.014$, Fig. 3 an instantaneous profile of the flame interface and Fig. 4 its temperature.

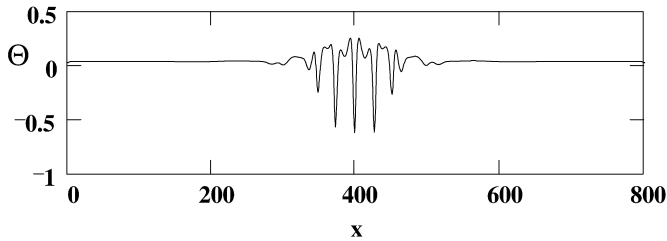


Fig. 4. Temperature distribution for the profile in Fig. 3.

5. Discussion

The offered $K-\Theta$ model (17), (18) is a product of successive coordinate-free extrapolations based on several special cases studied previously. For all that, the model is not merely a plausible suggestion. An independent perturbative analysis, assuming the background flow-field to be slowly varying both in space and time, yields largely the same results. The basic difference appears only in Θ terms of Eqs. (17), (18), which are now replaced by $\exp \Theta - 1$ and $1/4\Theta \exp \Theta$, respectively. When Θ is small, which is normally the case in adiabatic flames, both strategies lead to identical results, thereby vindicating the heuristic approach. Details of the perturbative derivation, extended from the 2D over the 3D flows, will be presented elsewhere.

The proposed $K-\Theta$ model is not limited to any particular range of Lewis numbers and is certainly applicable to diffusively stable flames where $Le > 1 - 2/\beta$.

If the Θ -terms are replaced by the above exponents the modified model is capable of covering the well-known effect of extinction of stretched planar flames when $Le > 1 + 2/\beta$. Here, as may be easily shown, one ends up with the following relation between Θ and the flow intensity q (Section 4),

$$\frac{1}{2}\beta q(1 - Le) = \Theta \exp \Theta. \quad (20)$$

At the extinction point $\Theta = -1$.

When the parameter $\alpha = 1/2\beta(1 - Le)$ approaches unity the cells become infinitely large. This difficulty of the constant density formulation is resolved by accounting for the burned gas thermal expansion, which keeps the cells finite at $\alpha = 1$ [13,14]. The density drop invokes the so-called hydrodynamic flame instability (Darrieus–Landau) triggered by the refraction of stream lines crossing the flame interface. More details on the Darrieus–Landau instability and its recent modeling may be found in Refs. [15–23].

The $K-\Theta$ model may be easily adapted for the variable density case. In this situation, as may be shown, the flow velocity \mathbf{U} in Eqs. (17), (18) should merely be replaced by \mathbf{U}_+ , the velocity at the downstream (burned) side of the interface. Unlike the constant density formulation, here the flow-field becomes affected by the flame configuration and cannot be regarded as prescribed. One thus ends up with a formidable hydrodynamic problem involving a free interface. Yet, in the limit of small density drop the problem is likely to become much more tractable. This should result in a modified Frankel-like model [15], applicable to general Lewis numbers and accounting for the simultaneous impact of the pre-existing and self-induced

flow-field. A systematic discussion of the variable density case will be presented elsewhere.

The $K-\Theta$ model (16), (17) and its various versions are not the only reduced models capable of mimicking the flame-flow interaction of diffusively unstable flames. Another possibility is suggested by the geometrically invariant extension of the KS equation [5,6] constituting a single fourth-order equation for the interface dynamics,

$$V_n = 1 - (\alpha - 1)\kappa - 4\kappa_{ss}. \quad (21)$$

Again, replacing V_n by $V_n - \mathbf{U} \cdot \mathbf{N}$, and κ by $-K$ one obtains,

$$V_n = \mathbf{U} \cdot \mathbf{N} + 1 + (\alpha - 1)K + 4K_{ss}. \quad (22)$$

Eq. (22), for all its appeal, is less general than the system (17), (18), and may prove to be less rich both structurally and dynamically. As mentioned earlier, the weakly nonlinear version of Eq. (22), applied to the stagnation-point flow [12], does not capture localization of the cellular structure clearly seen in Figs. 2–4.

Acknowledgements

The work of M.L.F. was supported in part by the NSF Grant DMS-0207308, P.G. was partially supported by the NSF Grant DMS-0554775, and G.I.S. was supported by the US–Israel Binational Science Foundation under Grant 2002-008, the Israel Science Foundation under Grants 350/05 and 278/03, and the European Community Program RTN-HPRN-CT-2002-00274.

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