

FINGERING INSTABILITY IN WATER-OIL DISPLACEMENT

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Abstract

Following the classical Buckley-Leverett theory for the two-phase immiscible flows in porous media a non-linear evolution equation for the water-oil displacement front is formulated and studied numerically. The numerical simulations yield a physically plausible picture of the fingering instability known to develop in water-oil systems. Distinctions and similarities with dynamically related Saffman-Taylor and Darrieus-Landau problems are outlined.

Introduction

In oil recovery technology it is common practice to inject water into the

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oil field at certain spots in an attempt to drive oil to certain other spots for pumping. In porous media of large extent the oil displacement by waterflooding often assumes the form of an advancing front, a moving zone of limited extent across which the saturation of water (the fraction of pore space occupied by water) changes rapidly compared to regions of nearly constant saturation upstream and downstream. In this process the phenomenon of fingering has long been identified. The water, which is intended to push the oil forward, tends to penetrate the oil through spontaneously formed multi-branched channels (fingers). As a result, the advancing front assumes a highly corrugated coral-like configuration (Wooding & Morel-Seytoux, 1976). The mechanism triggering the fingering in the two-phase flows is physically akin to that of the well-studied Saffman-Taylor (ST) problem dealing with the displacement of a viscous fluid by a less viscous one in a Hele-Shaw cell, a pair of parallel glass plates separated by a narrow gap (Homsy, 1987). In both systems the instability is caused by the high mobility of the displacing fluid, which results in similar dispersion relations in the long-wavelength range where the impact of capillary forces is negligible. The capillary forces provide relaxation of the short-wavelength disturbances which makes the associated evolution problem dynamically well-posed, that is free of pathological instabilities.

In the ST problem, accounting for the capillary forces is readily achieved by relating the pressure jump at the interface to its curvature and the surface tension (Laplace's law). For the displacement in two-phase flows the matter is more involved. Here the interplay between the flow-field and capillary forces is routinely described by the Buckley-Leverett theory where the saturation front acquires a finite-width structure. This feature, however, turns even the linear-stability analysis into a technically difficult problem which may be tackled only approximately or numerically (Barenblatt *et al.*, 1972, 1990; Hagoort, 1974; Yortsos & Huang, 1984; Jerault *et al.*, 1984 a,b; Chikhliwala & Yortsos, 1985). For all the different approaches employed there seems to

be a consensus that the marginal stability wavenumber k_c is proportional to the displacement rate (or the capillary number). Note that in the ST problem k_c is proportional to the square-root of the capillary number.

As has been shown in Barenblatt *et al.* (1972, 1990) for a slowly-varying front its normal velocity becomes a linear function of the front's curvature. This crucial observation, as discussed below, allows reduction of the original Buckley-Leverett formulation to a much more tractable free-interface problem where, similarly to the ST system, the capillary forces enter exclusively through the boundary conditions. Unlike the ST problem, however, here in the passage through the interface the pressure remains continuous. This distinction between the ST and two-phase systems is reflected in the different character of the pertinent relaxation rates ω , which affects the marginal stability wavenumber k_c .

In the ST problem $\omega \sim -k^3$ at $k \gg k_c$, whilst in the two-phase problem $\omega \sim -k^2$, as occurs in the classical diffusion equation. In the free-interface formulation the saturation ahead and behind the interface may be regarded as two prescribed constants S_a and S_b . As a result, the creeping flow-field on both sides of the interface becomes purely potential. This outcome allows reduction of the effective dimensionality of the system, yielding an explicit equation for the evolving interface, highly advantageous both for physical analysis and numerical simulations. The above ideas are by no means new and have, in various forms, been explored in modeling of different pattern-forming systems involving unstable interfaces (Bensimon *et al.*, 1986; Frankel, 1990; Kessler *et al.*, 1988; Langer, 1987; Pelce, 1988; Sivashinsky, 2002).

Yet, it seems that hitherto this approach has not been tried for the two-phase immiscible displacement in porous media, the topic of the present study.

The Buckley-Leverett model

The incompressible flow of two immiscible fluids in a porous medium can be described by the following equations (see e.g. Barenblatt *et al.*, 1990).

$$\varphi S_t + \nabla \cdot (F(S)\mathbf{u}) = \nabla \cdot (D(S)\nabla S) , \quad (1)$$

$$\mathbf{u} = -\lambda(S)\nabla P , \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

Here S is the saturation of the wetting (water) phase (fraction of water in the two-phase water-oil flow).

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}_o \quad (4)$$

is the total velocity of the two-phase flow. The subscript (w) and (o) denote water and oil phases, respectively.

$$P = p_w F(S) + p_o (1 - F(S)) - \int_S^1 p_c(\hat{S}) F'_s(\hat{S}) d\hat{S} \quad (5)$$

is the mean pressure of the two-phase flow. p_w and p_o are the pressures of the phases.

$$p_c(S) = p_o - p_w \quad (6)$$

is the capillary-pressure.

$$p_c(S) = \sigma \sqrt{\frac{\varphi}{K}} J(S) , \quad (7)$$

where σ is the surface tension, φ and K are the porosity and permeability of the porous medium, respectively. $J(S)$ is known as the Leverett function.

$$F(S) = \frac{\mu_o}{k_{ro}(S)} \left(\frac{\mu_w}{k_{rw}(S)} + \frac{\mu_o}{k_{ro}(S)} \right)^{-1} \quad (8)$$

is the fractional flow function, where μ_w, μ_o are the dynamic viscosities and k_{rw}, k_{ro} are the relative permeabilities of the phases.

$$\lambda(S) = -K \left(\frac{k_{rw}(S)}{\mu_w} + \frac{k_{ro}(S)}{\mu_o} \right) \quad (9)$$

is the total mobility of the two-phase flow.

$$D(S) = -K \frac{dp_c}{dS} \left(\frac{\mu_w}{k_{rw}(S)} + \frac{\mu_o}{k_{ro}(S)} \right)^{-1} \quad (10)$$

is the capillary-diffusivity. Equation (??) is known as the Buckley-Leverett equation, equation (??) is Darcy's law, and equation (??) expresses incompressibility.

Radial displacement

For further developments it is helpful to consider the case of a two-dimensional flow sustained by the point source. In this geometry the Buckley-Leverett model clearly allows for the purely radial solution with

$$\mathbf{u} = \frac{Q}{2\pi} \frac{\mathbf{r}}{|\mathbf{r}|^2}, \quad (11)$$

where Q is the source intensity, which might be either constant or time-dependent.

The Buckley-Leverett equation (??) thus becomes

$$\varphi \frac{\partial S}{\partial t} + \frac{Q}{2\pi r} \frac{\partial F(S)}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r D(S) \frac{\partial S}{\partial r} \right). \quad (12)$$

The saturation front, localized around some $r = R(t)$, is basically advected by the flow (??). Hence,

$$\frac{dR}{dt} \sim \frac{Q}{R}. \quad (13)$$

The front width d may therefore be defined as

$$d = D/R_t \sim DR/Q. \quad (14)$$

Thus, at $Q \gg D$ the front may be regarded as thin compared to its overall scale R .

In the frame of reference, $x = r - R(t)$, attached to the advancing front, equation (??) becomes,

$$\begin{aligned} \varphi \frac{\partial S}{\partial t} - \varphi \frac{dR}{dt} \frac{\partial S}{\partial x} + \frac{Q}{2\pi(R+x)} \frac{\partial F(S)}{\partial x} = \\ = \frac{D(S)}{R+x} \frac{\partial S}{\partial x} + \frac{\partial}{\partial x} \left(D(S) \frac{\partial S}{\partial x} \right) . \end{aligned} \quad (15)$$

Assuming the front structure to be quasi-steady, $x \sim d$, and $d \ll R$, equation (??) simplifies to,

$$\begin{aligned} -\varphi \frac{dR}{dt} \frac{\partial S}{\partial x} + \frac{Q}{2\pi R} \frac{\partial F(S)}{\partial x} = \\ -\frac{D(S)}{R} \frac{\partial S}{\partial x} + \frac{\partial}{\partial x} \left(D(S) \frac{\partial S}{\partial x} \right) . \end{aligned} \quad (16)$$

The global integration of equation (??) across the front then yields,

$$\varphi \frac{dR}{dt} = \frac{Q}{2\pi R} \frac{F_b - F_a}{S_b - S_a} - \frac{1}{R} \frac{G_b - G_a}{S_b - S_a} , \quad (17)$$

where

$$G(S) = - \int_S^1 D(\hat{S}) d\hat{S} , \quad (18)$$

and

$$S_b = S(-\infty), \quad S_a = S(+\infty) . \quad (19)$$

S_a, S_b is the water saturation ahead and behind the front, ($S_a < S_b$);

$$F_a = F(S_a), \quad F_b = F(S_b), \quad G_a = G(S_a), \quad G_b = G(S_b) .$$

In this, outer to the front's structure domain, the mean pressure is described by the equations,

$$\begin{aligned} \frac{Q}{2\pi r} = -\lambda_a \frac{\partial P}{\partial r} , \quad \lambda_a = \lambda(S_a) , \quad r > R , \\ \frac{Q}{2\pi r} = -\lambda_b \frac{\partial P}{\partial r} , \quad \lambda_b = \lambda(S_b) , \quad 0 < r < R , \end{aligned} \quad (20)$$

with P being continuous at $r = R$.

As has been shown by Ryzhik *et al.* (1961) (see also Barenblatt *et al.*, 1990) for the relation (??) to be meaningful the line between the points (S_a, F_a) and (S_b, F_b) should not intersect the fractional flow curve $F(S)$ between S_a and S_b . For a given saturation S_a this condition defines the range of speeds allowed for sustaining the saturation wave of a permanent structure.

Free-interface formulation

The found approximate solution (??)(?) for the radial displacement suggests the following geometrically-invariant formulation, applicable to general flow-field \mathbf{u} ,

$$\varphi \mathbf{n} \cdot \frac{d\mathbf{r}}{dt} = \frac{F_b - F_a}{S_b - S_a} u_n - \frac{G_b - G_a}{S_b - S_a} \kappa . \quad (21)$$

Here \mathbf{r} is a point on the evolving front \mathcal{L} ; \mathbf{n} is the outer normal to \mathcal{L} ; $\mathbf{n} \cdot d\mathbf{r}/dt = \nu_n$ is the normal velocity of \mathcal{L} ; $u_n = \mathbf{u} \cdot \mathbf{n}$ is the normal flow velocity at \mathcal{L} ; $\kappa = \nabla \cdot \mathbf{n}$ is the curvature.

$$\mathbf{u} = -\lambda_a \nabla P , \quad \mathbf{u} = -\lambda_b \nabla P . \quad (22)$$

ahead and behind the front, with P and \mathbf{u} remaining continuous. Equations (??)(?) should be considered jointly with the incompressibility condition (??).

Evaluation of u_n

By virtue of (??) and (??) the pressure beyond the front is described by the Laplace equation,

$$\nabla^2 P = 0 , \quad (23)$$

which should be considered jointly with the jump conditions on the front,

$$[\nabla P \cdot \mathbf{n}]_b^a = (\lambda_b^{-1} - \lambda_a^{-1}) \mathbf{u} , \quad (24)$$

$$[P]_b^a = 0 . \quad (25)$$

A further analysis of the system may therefore be conducted in the framework of the classical theory of the Newtonian or logarithmic potentials (see e.g. Koshlyakov *et al.*,1964), allowing to connect the shape of the interface with the normal velocity u_n appearing in (??).

In this paper the analysis is limited to the most simple situation where the saturation front is a closed curve \mathcal{L} evolving through a two-dimensional flow sustained by a point-source of a prescribed intensity Q . The solution of the problem defined by (??)(??)(??) then reads,

$$P(\mathbf{r}, t) = \frac{Q}{2\pi\lambda_b} \ell n \frac{1}{|\mathbf{r}|} - \left(\frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right) \frac{1}{2\pi} \int_{\mathcal{L}} u_n(\boldsymbol{\xi}, t) \ell n \frac{1}{|\mathbf{r} - \boldsymbol{\xi}|} d\mathcal{L}_{\boldsymbol{\xi}} . \quad (26)$$

The first term on the right of (??) stems from the point source located at $\mathbf{r} = 0$. On the b -side of the front,

$$\begin{aligned} \left(\frac{dP}{dn} \right)_n &= -\frac{Q}{2\pi\lambda_b} \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}|^2} + \frac{1}{2} \left(\frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right) u_n(\mathbf{r}, t) - \\ &- \frac{1}{2\pi} \left(\frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right) \int_{\mathcal{L}} \frac{u_n(\boldsymbol{\xi}, t) (\mathbf{r} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{r})}{|\mathbf{r} - \boldsymbol{\xi}|^2} d\mathcal{L}_{\boldsymbol{\xi}} . \end{aligned} \quad (27)$$

The pertinent calculations may be found, for example, in Koshlyakov *et al.* (1964). According to (??),

$$\left(\frac{dP}{dn} \right)_b = -\frac{1}{\lambda_b} u_n . \quad (28)$$

Inserting (??) into (??) yields,

$$\begin{aligned} \frac{Q}{2\pi\lambda_b} \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}|^2} &= \frac{1}{2} \left(\frac{1}{\lambda_a} + \frac{1}{\lambda_b} \right) u_n(\mathbf{r}, t) - \\ &- \frac{1}{2\pi} \left(\frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right) \int_{\mathcal{L}} \frac{u_n(\boldsymbol{\xi}, t) (\mathbf{r} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{r})}{|\mathbf{r} - \boldsymbol{\xi}|^2} d\mathcal{L}_{\boldsymbol{\xi}} . \end{aligned} \quad (29)$$

Equations (??)(??) fully determine the dynamics of the interface.

Linear stability analysis

For the radial solution, when \mathcal{L} is a circle, $r = R(t)$, the equation (??) readily yields,

$$u_n = \frac{Q}{2\pi R} . \quad (30)$$

For a slightly perturbed circular interface,

$$r = R(t) + \delta R(\theta, t) \quad (31)$$

where δR is the perturbation and θ is the polar angle. δR may be sought as,

$$\delta R = \sum_m A_m(t) \exp(im\theta) , \quad (32)$$

where m is the azimuthal wave-number and A_m is the corresponding amplitude. Then for the perturbations of the normal propagation velocity $\delta(\mathbf{n} \cdot \mathbf{r}_t)$ normal flow velocity δu_n and curvature $\delta\kappa$ one obtains,

$$\delta(\mathbf{n} \cdot \mathbf{r}_t) = \frac{dA_m}{dt} e^{im\theta} , \quad (33)$$

$$\delta u_n = \frac{Q}{2\pi R^2} \left[m \left(\frac{\lambda_b - \lambda_a}{\lambda_b + \lambda_a} \right) - 1 \right] A_m e^{im\theta} , \quad (34)$$

$$\delta\kappa = \frac{m^2 - 1}{R^2} A_m e^{im\theta} . \quad (35)$$

As a result, the perturbed version of (??) yields the following equation for A_m ,

$$\begin{aligned} \varphi \frac{dA_m}{dt} = & \left\{ \frac{Q}{2\pi R^2} \left(\frac{F_b - F_a}{S_b - S_a} \right) \left[m \left(\frac{\lambda_b - \lambda_a}{\lambda_b + \lambda_a} \right) - 1 \right] - \right. \\ & \left. - \left(\frac{G_b - G_a}{S_b - S_a} \right) \frac{m^2 - 1}{R^2} \right\} A_m . \end{aligned} \quad (36)$$

As one might expect, at a relatively high mobility of the displacing fluid ($\lambda_b > \lambda_a$) the radial flow becomes unstable. Moreover, there is a wave-number corresponding to the maximum growth rate of the perturbation.

Note again that due to the quasi-steady nature of the momentum equations (??) the source intensity Q may be either constant or time-dependent.

Scaling

To reduce the number of parameters involved the following non-dimensional variables and parameters are introduced,

$$\begin{aligned} \frac{\mathbf{r}}{r_{ref}} = \tilde{\mathbf{r}}, \quad \frac{t}{t_{ref}} = \tilde{t}, \quad \frac{u_n}{u_{ref}} = \tilde{u}_n, \\ \frac{Q}{r_{ref}u_{ref}} = q, \quad \nu = \frac{\lambda_a}{\lambda_b}, \quad \gamma = \frac{1}{r_{ref}u_{ref}} \frac{G_b - G_a}{F_b - F_a}. \end{aligned} \quad (37)$$

Here $r_{ref}, t_{ref}, u_{ref}$ are the reference scales t_{ref} is expressed in terms of r_{ref} and u_{ref} as

$$t_{ref} = \varphi \frac{r_{ref}}{u_{ref}} \frac{S_b - S_a}{F_b - F_a}. \quad (38)$$

Parameters ν, γ and q may be regarded as relative mobility, scaled capillary-diffusivity, and source intensity, respectively.

Omitting ‘tildes’ the scaled version of the model (??)(??) reads,

$$\mathbf{n} \cdot \frac{d\mathbf{r}}{dt} = u_n - \gamma\kappa, \quad (39)$$

$$\begin{aligned} \frac{\nu q \mathbf{r} \cdot \mathbf{n}}{2\pi |\mathbf{r}|^2} = \frac{1}{2}(1 + \nu)u_n(\mathbf{r}, t) - \\ - \frac{1}{2\pi}(1 - \nu) \int_{\mathcal{L}} \frac{u_n(\boldsymbol{\xi}, t)(\mathbf{r} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{r})}{|\mathbf{r} - \boldsymbol{\xi}|^2} d\mathcal{L}_{\boldsymbol{\xi}}. \end{aligned} \quad (40)$$

For the constant source intensity, considered below, u_{ref} may be defined as,

$$u_{ref} = Q/r_{ref} \quad (q = 1), \quad (41)$$

and r_{ref} as the length-scale of the initial configuration. There is no intrinsic length-scale in this case.

Some geometrical relations

In this section some useful relations from differential geometry are presented which are needed for the subsequent numerical simulations. More details may

be found in Brower *et al.* (1984) and Langer (1987). It is common practice to parameterize the evolving front by its arclength s i.e. to represent \mathbf{r} as,

$$\mathbf{r} = (x(s, t), y(s, t)) \quad (42)$$

where

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1 \quad (43)$$

hence,

$$\mathbf{n} = \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s}\right). \quad (44)$$

By virtue of (??) one may set,

$$\frac{\partial x}{\partial s} = -\sin \Theta(s, t), \quad \frac{\partial y}{\partial s} = \cos \Theta(s, t), \quad (45)$$

where Θ is the angle between the normal to the front and some fixed direction, say, the x -axis.

The normal advancement of the front at the rate $\mathbf{n} \cdot d\mathbf{r}/dt = \nu_n$ is automatically ensured if one sets,

$$\left(\frac{\partial x}{\partial t}\right)_n = \nu_n \cos \Theta, \quad \left(\frac{\partial y}{\partial t}\right)_n = \nu_n \sin \Theta. \quad (46)$$

The subscript n denotes temporal differentiation along the outward normal to the front. Upon spatial differentiation equations (??) yield,

$$\begin{aligned} \left(\frac{\partial}{\partial t} \left(\frac{\partial x}{\partial s}\right)\right)_n &= \frac{\partial \nu_n}{\partial s} \cos \Theta - \nu_n \sin \Theta \frac{\partial \Theta}{\partial s}, \\ \left(\frac{\partial}{\partial t} \left(\frac{\partial y}{\partial s}\right)\right)_n &= \frac{\partial \nu_n}{\partial s} \sin \Theta + \nu_n \cos \Theta \frac{\partial \Theta}{\partial s}, \end{aligned} \quad (47)$$

or by (??),

$$\begin{aligned} -\cos \Theta \left(\frac{\partial \Theta}{\partial t}\right)_n &= \frac{\partial \nu_n}{\partial s} \cos \Theta - \nu_n \sin \Theta \frac{\partial \Theta}{\partial s}, \\ -\sin \Theta \left(\frac{\partial \Theta}{\partial t}\right)_n &= \frac{\partial \nu_n}{\partial s} \sin \Theta + \nu_n \cos \Theta \frac{\partial \Theta}{\partial s}. \end{aligned} \quad (48)$$

Eliminating $\partial\Theta/\partial s$ one ends up with the well-known relation,

$$\left(\frac{\partial\Theta}{\partial t}\right)_n = -\frac{\partial\nu_n}{\partial s} . \quad (49)$$

There is a simple connection between the normal time-derivative and partial derivative at constant s (Langer, 1987),

$$\left(\frac{\partial\Theta}{\partial t}\right)_n = \frac{\partial\Theta}{\partial t} + \left(\frac{\partial s}{\partial t}\right)_n \frac{\partial\Theta}{\partial s} , \quad (50)$$

where

$$\left(\frac{\partial s}{\partial t}\right)_n = \int_0^s \kappa\nu_n d\hat{s} \quad (51)$$

Employing (??)(??), the equation (??) transforms to,

$$\frac{\partial\Theta}{\partial t} = -\frac{\partial\nu_n}{\partial s} - \left(\int_0^s \kappa\nu_n d\hat{s}\right) \frac{\partial\Theta}{\partial s} . \quad (52)$$

Note that in terms of Θ the local curvature κ becomes,

$$\kappa = \frac{\partial^2 y}{\partial s^2} \frac{\partial x}{\partial s} - \frac{\partial^2 x}{\partial s^2} \frac{\partial y}{\partial s} = \frac{\partial\Theta}{\partial s} . \quad (53)$$

In numerical simulations it is convenient to deal with the normalized arclength,

$$\alpha = s/L(t) , \quad (54)$$

where L is the total arclength of the evolving front. In terms of α equation (??) becomes,

$$\frac{\partial\Theta}{\partial t} = -\frac{1}{L} \frac{\partial\nu_n}{\partial\alpha} - \left(\int_0^\alpha \kappa\nu_n d\hat{\alpha} - \alpha \int_0^1 \kappa\nu_n d\hat{\alpha}\right) \frac{\alpha\Theta}{\partial\alpha} . \quad (55)$$

Equation (??) applied to the total arclength L yields,

$$\frac{dL}{dt} = L \int_0^1 \kappa\nu_n d\hat{\alpha} , \quad (56)$$

which is a necessary supplement to equation (??).

Equations (??)(??) are purely geometrical statements valid for any ν_n . The front dynamics is specified by the relation,

$$\nu_n = u_n - \gamma\kappa \quad (\text{see (39)}) . \quad (57)$$

where u_n is defined by (??). In terms of the normalized arclength the latter reads,

$$\begin{aligned} \frac{\nu}{2\pi} \frac{1}{L} \frac{xy'_\alpha - yx'_\alpha}{x^2 + y^2} &= \frac{1 + \nu}{2} u_n + \\ &+ \frac{1 - \nu}{2\pi} \int_0^1 \mathcal{H}(\alpha, \hat{\alpha}, t) u_n(\hat{\alpha}, t) d\hat{\alpha} . \end{aligned} \quad (58)$$

where

$$\mathcal{H}(\alpha, \hat{\alpha}, t) = \frac{x'_\alpha [y(\alpha, t) - y(\hat{\alpha}, t)] - y'_\alpha [x(\alpha, t) - x(\hat{\alpha}, t)]}{[x(\alpha, t) - x(\hat{\alpha}, t)]^2 + [y(\alpha, t) - y(\hat{\alpha}, t)]^2} \quad (59)$$

There is no singularity at $\alpha = \hat{\alpha}$. As may be easily checked,

$$\mathcal{H}(\alpha, \alpha, t) = -\frac{1}{2} L(t) \kappa(\alpha, t) . \quad (60)$$

The coordinate functions $x(\alpha, t)$, $y(\alpha, t)$ are governed by the transformed versions of (??)(??), which read,

$$\frac{\partial x}{\partial \alpha} = -L \sin \Theta , \quad \frac{\partial y}{\partial \alpha} = L \cos \Theta , \quad (61)$$

$$\frac{\partial x}{\partial t} = \nu_n \cos \Theta - \left(\int_0^\alpha \kappa \nu_n d\hat{\alpha} - \alpha \int_0^1 \kappa \nu_n d\hat{\alpha} \right) \frac{\partial x}{\partial \alpha} , \quad (62)$$

$$\frac{\partial y}{\partial t} = \nu_n \sin \Theta - \left(\int_0^\alpha \kappa \nu_n d\hat{\alpha} - \alpha \int_0^1 \kappa \nu_n d\hat{\alpha} \right) \frac{\partial y}{\partial \alpha} .$$

Equations (??)–(??) are considered jointly with the following set of boundary and initial conditions,

$$\Theta(1, t) = \Theta(0, t) + 2\pi , \quad (63)$$

$$\Theta'_\alpha(1, t) = \Theta'_\alpha(0, t) , \quad (64)$$

$$x(1, t) = x(0, t), \quad y(1, t) = y(0, t) , \quad (65)$$

$$\Theta(\alpha, 0) = \Theta_0(\alpha) , \quad (66)$$

$$L(0) = L_0 . \quad (67)$$

Numerical strategy

The quasi-steady nature of the equation (??) prevents evaluation of $u_n(\alpha, t)$ as a solution of an initial-value problem. Yet, it appears that $u_n(\alpha, t)$ can be found by an iterative procedure, considering (??) as a limit of the recursive process,

$$u_n^{(k+1)}(\alpha, t) = \frac{\nu}{\pi(1+\nu)} \frac{1}{L} \frac{xy'_\alpha - yx'_\alpha}{x^2 + y^2} - \frac{1}{\pi} \frac{1-\nu}{1+\nu} \int_0^1 \mathcal{H}(\alpha, \hat{\alpha}, t) u_n^{(k)}(\hat{\alpha}, t) d\hat{\alpha} . \quad (68)$$

Here the superscript (k) labels the iteration number; $k = 0, 1, 2, \dots, k_{end}$, where k_{end} is defined by the requirement,

$$\max_{\alpha} |u_n^{(k_{end})}(\alpha, t) - u_n^{(k_{end}-1)}(\alpha, t)| < \delta(t) , \quad (69)$$

where $\delta(t)$ is the prescribed accuracy set as,

$$\delta(0) = 10^{-10} \quad \text{and} \quad \delta(t > 0) = 10^{-5} \quad (70)$$

In (??) the zeroth approximations are specified as,

$$\begin{aligned} u_n^{(0)}(\alpha, 0) &= 1/L_0 \quad \text{for} \quad t = 0, \quad \text{and} \\ u_n^{(0)}(\alpha, t) &= u_n^{(k_{end})}(\alpha, t - \Delta t) \quad \text{for} \quad t > 0 . \end{aligned} \quad (71)$$

Here $1/L_0$ corresponds to the radial flow, and Δt is the time-step.

The derivatives x'_α, y'_α appearing in (??) are defined by equations (??). Evaluation of the coordinate functions $x(\alpha, t)$, $y(\alpha, t)$ however, is a more subtle matter. The point is, that by virtue of the boundary conditions (??), equations (??) yield,

$$\int_0^1 \sin \Theta(\alpha, t) d\alpha = \int_0^1 \cos \Theta(\alpha, t) d\alpha = 0 . \quad (72)$$

In numerical simulations, due to the unavoidable accumulation of errors, the conditions (??) are easily violated. As a result, the ends of the curve \mathcal{L} might

not meet, creating a physically unacceptable situation. To circumvent this difficulty, $x(\alpha, t)$, $y(\alpha, t)$ are determined from the second-order equations

$$\begin{aligned}\frac{\partial^2 x}{\partial \alpha^2} &= -L \cos \Theta \frac{\partial \Theta}{\partial \alpha} , \\ \frac{\partial^2 y}{\partial \alpha^2} &= -L \sin \Theta \frac{\partial \Theta}{\partial \alpha} ,\end{aligned}\tag{73}$$

obtained by differentiation of (??), and considered jointly with the boundary conditions, provided by equations (??),

$$\begin{aligned}x(0, t) &= \int_0^t \cos \Theta(0, \hat{t}) \nu_n(0, \hat{t}) d\hat{t} + x(0, 0) , \\ x(1, t) &= \int_0^t \cos \Theta(1, \hat{t}) \nu_n(1, \hat{t}) d\hat{t} + x(1, 0) , \\ y(0, t) &= \int_0^t \sin \Theta(0, \hat{t}) \nu_n(0, \hat{t}) d\hat{t} + y(0, 0) , \\ y(1, t) &= \int_0^t \sin \Theta(1, \hat{t}) \nu_n(1, \hat{t}) d\hat{t} + y(1, 0) ,\end{aligned}\tag{74}$$

According to (??),

$$\begin{aligned}x(0, 0) &= x(1, 0) = x_{00} , \\ y(0, 0) &= y(1, 0) = y_{00} ,\end{aligned}\tag{75}$$

where x_{00} , y_{00} are specified by the initial configuration.

The advantage of equations (??) is that, due to (??), they are always compatible with the boundary conditions (??)(??), whatever the profile $\Theta(\alpha, t)$ might be.

The boundary-value problems (??)–(??) are solved by replacing the second-order derivatives by the appropriate finite-differences (symmetric, of the second-order accuracy). The resulting 3-diagonal matrix is inverted by the conventional forward-backward-marching technique (see e.g. Godunov & Ryabenki, 1964).

The dynamical equation (??) is solved by the implicit (predictor-corrector) time-stepping.

In order not to single-out the end points ($\alpha = 0$, $\alpha = 1$) the original boundary conditions (??)(??) are replaced by the periodic conditions,

$$\begin{aligned}\Theta(1 - \Delta\alpha, t) &= \Theta(-\Delta\alpha, t) + 2\pi, \\ \Theta(1 - t) &= \Theta(0, t) + 2\pi,\end{aligned}\tag{76}$$

where $\Delta\alpha$ is the mesh-step.

The spatial derivatives are approximated by symmetric, second-order-accuracy finite-differences. On each time-step the associated boundary-value problem is again solved by the forward-backward-marching, modified for the periodic conditions (??).

Numerical simulations

Figures 1*a, b, c, d* present results of the numerical simulation of the model (??)-(??) conducted for $\nu = 0.15$ (*a*), 0.015(*b, c*), 0.0015(*d*), $\gamma = 0.001$, $q = 1$, $L_0 = 2\pi$, $x_{00} = 1$, $y_{00} = 1$, $\Theta_0(\alpha) = 2\pi\alpha + \varepsilon \sin(12\pi\alpha)$, with $\varepsilon = -0.05$ (*a, b*) and $\varepsilon = 0.05$ (*c, d*).

The spatio-temporal steps are set as $\Delta\alpha = 1/2500 = 0.0004$ and $\Delta t = 0.00125$. For the accuracy defined by (??) the number of the required iterations comes to $k_{end} = 30$ for $t = 0$, and $k_{end} \leq 10$ for $t > 0$.

A small perturbation imposed on a circular front, results in a rapid development of permanently growing primary and secondary fingers (Figures 1*b, c, d*). The latter are likely to undergo further fractalization leading to a highly convoluted coral-like interface. Such an intricate behavior, though consistent with observations, is still rather puzzling. One would expect, that in the absence of the intrinsic length-scale, the long-time evolution should result in a self-similarly growing star-shaped configuration. It has long been argued, in the context of related pattern forming systems (Bensimon *et al.*, 1986; Kessler *et al.*, 1988; Joulin & Vidal, 1996), that the complex dynamics might well be the system's response to an ever present background (e.g. numerical) noise. In the intrinsically stable situation ($\nu > 1$) the impact of noise

is too insignificant to cause a noticeable distortion of the advancing front. However, at $\nu < 1$ the noise may play the important role of a permanently acting trigger supplying small, but finite, disturbances rapidly magnified by the intrinsic instability. As could be anticipated, near the stability threshold ($\nu \lesssim 1$, Figure 1b) the fingering pattern appears considerably more regular than the pattern developing deep inside the instability domain ($\nu \ll 1$, Figures 1b, c, d). A marked sensitivity of the dynamical response to the sign of ε (Figures 1b,c) is explained as follows: for the chosen initial profile $\Theta_0(\alpha)$, as implied by (??), there is a small gap ($\sim \varepsilon$) between the ends of the curve $x(\alpha, 0)$, $y(\alpha, 0)$, $0 \leq \alpha \leq 1$. Although this difficulty is successfully resolved by the higher-order formulation (??)–(??), the modified (closed) curve still suffers a small kink as it crosses the x -axis. For $\varepsilon = 0.05$ the kink falls on the apex of the initial ‘hexagon’, whilst for $\varepsilon = -0.05$ it appears in the middle of the hexagon’s side. This distinction in the initial configurations (aside from the 30° – turn) results in a progressively different character of the subsequent evolution.

Concluding remarks

One of the difficulties in numerical simulations of the reduced model (??) (??) is the extraction of u_n from the time-independent equation (??). At small ν this turns into a a very slow procedure. Yet, at $\nu \simeq 1$ (the case of light oils) the recursive process (??) may be truncated upon the very first iteration, taking the radial flow as the zeroth approximation. Hence, one readily obtains,

$$u_n(\mathbf{r}, t) = w_n(\mathbf{r}, t) + \frac{1 - \nu}{2\nu} \frac{1}{\pi} \int_{\mathcal{L}} \frac{w_n(\boldsymbol{\xi}, t) (\mathbf{r} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{r})}{|\mathbf{r} - \boldsymbol{\xi}|^2} d\mathcal{L}_\xi, \quad (77)$$

where

$$w_n(\mathbf{r}, t) = \frac{q\nu}{\pi(1 + \nu)} \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}|^2}. \quad (78)$$

The factor $(1 - \nu)/2\nu$ in (??) is chosen to preserve the radial solution ($u_n = q/2\pi R$) pertinent to the circular front ($r = R$). The new model(??)(??)(??) is clearly much more benign computationally than the original. Its numerical exploration, however, is beyond the scope of the present study, and will be reported elsewhere.

It is curious that for $w_n = \text{const}$ the model based on the equations (??)(??) is formally identical to Frankel's (1990) equation for the premixed gas flames subjected to the Darrieus-Landau instability (see e.g. Sivashinsky, 1983). There is, however, an important dynamical distinction between flames and displacement fronts. Unlike the displacement front driven by the background flow-field the flame spread is a self-sustaining process driven by the interplay between chemical heat release and molecular transport. Here w_n does not depend on the background flow, and might be regarded as a physico-chemical parameter of the system. The constancy of w_n brings about a strong, purely geometrical, mechanism of non-linear stabilization (Sivashinsky 1983), missing in the displacement problem. As a result, the flame instability manifests itself in the formation of a shallow wrinkled structure (cells) rather than prominent protuberances (fingers) typical of the displacement fronts.

As has been mentioned in the Introduction, there is an essential difference between the ST and two-phase displacements expressed in how the capillary forces affect the process. Unlike the two-phase system, in the ST displacement ν_n does not depend on the curvature directly, but rather on its variation along the interface. As a result, in the two-phase system the small-scale relaxation is governed by the local term $\sim \kappa$, whereas in the ST system it is a non-local effect. In this sense the two-phase system, though being structurally more complex, transpires to be considerably simpler dynamically. For all this distinction, the character of the developing patterns in both systems seems to be quite similar qualitatively. For identical fluids the impact of the capillary forces in the two-phase problem, however, is likely to be weaker,

which might result in a more convoluted fingering pattern. This question, however, requires further research.

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Figure captions

Figure 1. Displacement front at several consecutive equidistant instants of time. Numerical simulations of the model (??)–(??) for $\nu = 0.15(a)$, $0.015(b, c)$, $0.0015(d)$, $\gamma = 0.001$ and $q = 1$.

The time-interval between the plotted curves is set at 3, or $3/0.00125 = 2400$ time-steps.