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Dynamics of Thermally-Insulated Nonequilibrium Stefan Problem

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Abstract. We study a two-phase Stefan problem with kinetics. Here we prove existence of a finite-dimensional attractor for the problem without heat losses. Fot the most part we use a more elegant technique of energetic type estimates in appropriately defined weighted Sobolev spaces as opposite to the parabolic potentials of [9]. We demonstrate existence of compact attractors in the Sobolev spaces and prove that the attractor consists of sufficiently regular functions. This allows us to show that the Hausdorff dimension of the attractor is finite.

1. Introduction

The principal result of the present paper is the proof of existence of a finite-dimensional compact attractor for the two-phase Stefan problem with kinetic condition *without heat losses*. This Free-Interface Problem (FIP) is employed to model propagation of condensed phase combustion fronts and some phase transition interfaces (see, for example, [18, 11, 20]). The condensed phase combustion, also known as Combustion Synthesis finds technological applications in synthesis of some technologically advanced materials, see [15], [21] and also [22] for a popular exposition.

The FIP with kinetics is known to generate a variety of complex thermokinetic oscillations [7] such as Hopf bifurcation, period doubling cascades resulting in chaotic dynamics, a Shilnikov-Hopf bifurcation etc. In the appropriately nondimensionalized variables the problem is formulated as follows: find $s(t)$ and $u(x,t)$ such that

$$u_t = u_{xx}, \quad x \neq s(t), \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0, \tag{1.2}$$

$$g[u(s(t),t)] = v(t) < 0, \tag{1.3}$$

$$[u_x(s(t),t)] := u_x^+(s(t),t) - u_x^-(s(t),t) = v(t) \tag{1.4}$$

where $v(t)$ is the interface velocity, $s(t) = \int_0^t v(\tau)d\tau$ is its position, u is the temperature, and the one-sided derivatives u_x^+ and u_x^- are taken at the free interface. We would like to

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stress that, in contrast with our previous work [9], in the heat equation (1.1) the Newton's cooling term is absent.

The evolution of the model in (1.1)–(1.4) is governed by the interplay between the heat release due to the kinetics $g(u|_{x=s(t)})$ and the heat dissipation by the medium. The jump condition (1.4) (the Stefan boundary condition) reflects the balance between the heat production at the free boundary and its diffusion by the medium. The condition (1.3) is a manifestation of the *nonequilibrium* nature of the transition; for the classical Stefan problem it is replaced by the condition $u|_{x=s(t)} = 0$. We also require that the solution at minus infinity (far ahead of the advancing interface) should satisfy $\lim_{x\to-\infty} u(x,t) = 0$. This requirement is related to the front propagation in the negative x-direction ($v < 0$).

Numerical calculation of the correlation dimension based on direct numerical simulation of FIP (1.1)–(1.4) suggests that the essential dynamics of the free-interface problem should be finite-dimensional. Indeed, *in the present paper we prove that FIP (1.1)–(1.4) possesses a compact finite-dimensional attractor*. Using weighted Sobolev spaces directly, one can obtain compactness and finite dimensionality results without the additional damping in [9].

Next, we present an outline of the logical structure of the paper; a more detailed version of the paper is posted in [10]. In Sec. 2 we collect some pertinent background information. In Sec. 3 we state the main existence results of the paper (Theorem 3) in weighted Sobolev spaces H^1_ω with initial data in H_ω (see (3.7) for the definition). The proof of the theorem is quite lengthy and consists of several logical steps that are carried out in the Sections 4–6. First in Sec. 4 we prove existence of the interface velocity, and derive a global estimate for the interface temperature. Once the interface velocity is determined, one can construct a classical solution explicitly via the representation formula (4.12) below. The a priori energy estimates (5.22) in Sections 5–6 then demonstrate that the solution belongs to appropriate functional spaces and is bounded as stated in Theorem 3. This implies global existence. Even more importantly (5.22) demonstrates the existence of an absorbing set for the free-interface problem and therefore establishes dissipativity of the problem.

There is a substantial literature that treats basic existence and well-posedness issues for initial–boundary value problems for different sharp-interface models with kinetics, related to the problem (1.1)–(1.4), see, for example, [13, 16, 3, 4]. These works are mostly concerned with local in time existence. In recent papers Brauner *et al.* and Lorenzi, [1, 2, 12] study weakly-nonlinear dynamics for related problems. In particular they consider perturbations of traveling-wave initial data and investigate their instability and bifurcations. In contrast, the present paper focuses on *strongly nonlinear dynamics for a wide range of initial data and parametric regimes.*

In subsequent sections we obtain estimates, which allow us to prove compactness of the attractor later on. The solution of (1.1)–(1.4) can naturally be viewed as a sum of two parts: a contribution from the interface and such from the initial data. The estimate (5.27) shows exponential decay for the H_ω-norm of the contribution from the initial data, while (5.30) provides a uniform bound for the contribution from the interface. The additional ingredient in the proof of compactness of the attractor is presented in Sec. 5.2: the norms of the spatial "tails" of the interface contributions decay uniformly in time, (5.31)–(5.32).

In Sec. 6 we obtain uniform bounds on the H_ω-norm of the derivative (6.35). The bounds are derived through the potential theory representation, however we remark that one could also obtain them via energy estimates, combined with the Uniform Gronwall Lemma [19, Chap. 3]. The estimate in (6.35) is used in the compactness proof, as well as in the dimension estimate in Sec. 9 and 10.

In Sec. 7 we prove that solutions of the free boundary problem (1.1)–(1.4) depend continuously on initial data (Theorem 18); in other words we establish continuity of the corresponding evolution operator. This fact is a necessary prerequisite for the general theorems used in Sec. 8 to hold. Theorem 18 is also used for the proof of differentiability of the evolution operator in Sec. 10.

In Sec. 9 we estimate the Hausdorff dimension of the attractor based on the techniques described for instance in [19]. We study evolution of the infinitesimal volume along the trajectories in the attractor and demonstrate that for sufficiently large m that is defined solely by the physical properties of the problem, the m-dimensional volume decays exponentially. After that in Sec. 10 we prove that the evolution operator is uniformly differentiable with respect to the initial data. This, combined with the estimate for the linearized evolution of the infinitesimal volume, leads to the conclusion that the Hausdorff dimension of the attractor is finite.

We conclude with a remark on the notation convention: unless otherwise indicated, *all the spatial integrals are assumed to be on the whole real line*, i.e., we always write $\int \phi(x)dx$ instead of $\int_{-\infty}^{\infty} \phi(x)dx$.

2. Properties of solutions: Previous results

In this section we present some pertinent background information from [6, 8] (certain statements are slightly modified and clarified). The following theorem summarizes existence results:

THEOREM 1. *Let the kinetic functions g satisfy the following assumptions:*

(A1) $g(u)$ *is a continuously differentiable, negative function, with* $g'(u) < 0$ *on* $[0, \infty)$ *and* $g(0) = -v_0$ *for some velocity* $-v_0 < 0$

(A2) $g(u)$ *is sublinear:* $\lim_{u\to\infty} g(u)/u = 0$.

Let the initial data $u_0(x) \in C(-\infty, \infty)$. *Then there exists one and only one classical solution of the free interface problem (1.1)–(1.4). The solution is uniformly bounded for all* $t > 0$.

For the clarity of presentation, on many occasions in the sequel we replace the sublinearity condition $(A2)$ by a stronger condition: We will assume that $g(u)$ is a monotonically decreasing differentiable function on $[0, \infty]$ with $|g'| \leq L$ and satisfying

$$-V_0 \leq g(u) \leq -v_0 \text{ for some } V_0, v_0 > 0. \tag{2.5}$$

These conditions are satisfied, for instance, for the standard Arrhenius kinetics, (in the context of condensed state combustion see, for example, [7]). *However all the results of the paper hold for the less restrictive sublinear kinetics as well.*

Under the assumptions of Theorem 1 and some additional conditions the following smoothness result holds [6, 8]:

THEOREM 2. *Let the initial data u_0 be twice differentiable in $x < 0$ and $x > 0$ with bounded derivatives. Let u_0 satisfy the following matching condition:*

$$g(u_0(0)) = \frac{\partial u_0^+}{\partial x}(0) - \frac{\partial u_0^-}{\partial x}(0), \tag{2.6}$$

in addition, let the derivative of the kinetics function g' be Lipschitz continuous. Then the velocity v is differentiable.

3. Statement of the existence result

For our purposes we need the weighted Sobolev spaces H_ω, H^1_ω

$$\|f\|_\omega = \left(\int \omega(x)\,|f|^2\,dx\right)^{1/2}, \quad \|f\|_{\omega,1} = \|f\|_\omega + \|\partial_x f\|_\omega \tag{3.7}$$

with the weight $\omega(x) = e^{-\alpha x}$, $\alpha > 0$. The choice of the weight ω is dictated by the spectral properties of the linearization of (1.1)–(1.4). It allows one to shift into the negative real part half-plane the essential spectrum of the linearization about any solution from the attractor. This shift is also instrumental in the proof of a version of the Hopf theorem (cf. [17, 5]).

To follow the evolution in weighted spaces we recast the problem in the interface-attached coordinate frame. We introduce $\zeta = x - s(t)$, then the dependent variable becomes $\widetilde{u}(\zeta, t) = u(\zeta + s(t), t)$; with the customary abuse of notation we still use $u(\zeta, t)$ for $\widetilde{u}(\zeta, t)$. Then the problem takes the form:

$$u_t = u_{\zeta\zeta} + v(t)u_\zeta, \quad \zeta \neq 0, \quad t > 0, \tag{3.8}$$

$$g[u(0, t)] = v(t), \tag{3.9}$$

$$[u_\zeta(0, t)] := u_\zeta^+(0, t) - u_\zeta^-(0, t) = v(t), \tag{3.10}$$

$$u(\zeta, 0) = u_0(\zeta) \geq 0,$$

All the norm estimates in the subsequent sections should be understood in the ζt-coordinates. Alternatively the estimates can be carried out in the original coordinates, with the shifted weight function $\omega(x - s(t))$. Thus, for any F

$$\|F(.,t)\|_\omega := \left(\int |F(\zeta,t)|^2\,\omega(\zeta)d\zeta\right)^{1/2} = \left(\int |F(x,t)|^2\,\omega(x - s(t))dx\right)^{1/2}$$

The main existence result of the paper is the following global existence theorem for the free interface problem *with initial data in* H_ω (cf. Theorem 1.):

THEOREM 3. *Suppose that the kinetic function* $g(u)$ *is a continuously differentiable, monotone decreasing, negative function on* $[0,\infty)$ *with* $-V_0 \le g(u) \le -v_0$ *for some* $V_0, v_0 > 0$, *and* $|g'| \le L$ *for some* $L > 0$. *Suppose the initial data* $u_0 \in H_\omega$. *Then*

(i) *There exists one and only one solution of the free interface problem (1.1)–(1.4) such that* $u \in C([0,\infty), H_\omega) \cap C((0,\infty), H^1_\omega)$ *and* $v \in C(0,\infty)$.
(ii) *Moreover,* $\|u(.,t)\|_\omega \le C$ *for all* t, *while* $\|u(.,t)\|_{\omega,1} \le C_1$ *for* $t \ge t_0 > 0$, *where* C_1 *depends on* t_0.
(iii) *The solution is classical and uniformly bounded for all* $t \ge t_0 > 0$.

The proof of the theorem is quite lengthy and consists of several logical steps that are carried out in the Sections 4–6. First in Sec. 4 we prove existence of the interface velocity, and derive a global estimate for the interface temperature. Once the interface velocity is determined, one can construct a classical solution explicitly via the representation formula (4.12) below. The a priori energy estimates (5.22) in Sections 5–6 then demonstrate that the solution belongs to appropriate functional spaces and is bounded as stated in Theorem 3. This implies global existence.

4. Integral equation for the interface velocity

4.1. *Integral representation for solutions*

The proof of Theorem 3 utilizes the reduction to an integral equation for the interface velocity,

$$v(t) = g\left(\int G(s(t)-\xi, t)u_0(\xi)d\xi - \int_0^t G(s(t)-s(\tau), t-\tau)v(\tau)d\tau\right). \qquad (4.11)$$

The latter arises from the interface boundary condition, when the solution is sought via the single layer potential representation (see [8]):

$$\begin{aligned}(Tu_0)(\zeta,t) &= u(\zeta,t) = -\int_0^t G(\zeta + s(t) - s(\tau), t-\tau)v d\tau \\ &\quad + \int G(\zeta + s(t) - \xi, t)u_0 d\xi, \end{aligned} \qquad (4.12)$$

where

$$G(x-\xi,t-\tau)=\exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\}[4\pi(t-\tau)]^{-1/2},\, s(t)=\int_0^t v(\tau)d\tau.$$

It is convenient to view the representation formula (4.12) for the solution operator T as a sum of the two parts: the contribution of the free boundary

$$u_1(\zeta,t):=T_1(t)u_0(\zeta)=-\int_0^t G(\zeta+s(t)-s(\tau),t-\tau)v(\tau)d\tau \tag{4.13}$$

and that of the initial data

$$u_2(\zeta,t):=T_2(t)u_0(\zeta):=\int G(\zeta+s(t)-\xi,t)u_0(\xi)d\xi \tag{4.14}$$

4.2. *Local existence for interface velocity*

Let Kv denote the action of the right hand side of (4.11) with a fixed $u_0\in H_\omega$. Our first step is to show that K is a contraction.

PROPOSITION 4. *The transformation* $\psi=K\varphi$ *is a contraction on a closed subset* E_σ *of* $C(0,\sigma]$ *for some* σ, $E_\sigma:=\{v: v_0\le -v(t)\le V_0, 0<t\le\sigma\}$ *for some* $\sigma>0$. *Therefore* K *has a unique fixed point* $v=Kv$.

Proof. We will use the notation $|\phi|_\sigma:=\sup_{0<t\le\sigma}|\phi(t)|$. First we note that K maps E_σ into itself, since the range of the function g is $[-V_0,-v_0]$. Now, let $\psi=K\varphi$ and $\psi'=K\varphi'$, then

$$\begin{aligned}|\psi-\psi'| &\le L\left|\int[G(s-\xi,t)-G(s'-\xi,t)]u_0(\xi)d\xi\right.\\ &\qquad\left.+\int_0^t[G(s(t)-s(\tau),t-\tau)\varphi-G(s'(t)-s'(\tau),t-\tau)\varphi']d\tau\right|\\ &= L\left|\int[G(s-\xi,t)-G(s'-\xi,t)]u_0(\xi)d\xi+\int_0^t\Delta G\varphi'(\tau)d\tau\right.\\ &\qquad\left.+\int_0^t G(s(t)-s(\tau),t-\tau)\left(\varphi-\varphi'\right)d\tau\right|:=L\,|W_1+W_2+W_3|\end{aligned} \tag{4.15}$$

where L is the Lipschitz constant for g, $s(t)=\int_0^t\varphi d\tau$, $s'(t)=\int_0^t\varphi' d\tau$, and

$$|\Delta G|:=|G(s(t)-s(\tau),t-\tau)-G(s'(t)-s'(\tau),t-\tau)|$$

First, by the mean value theorem $|\Delta G|=\frac{1}{2}|\varphi(\tau')-\varphi'(\tau')|\,|\tilde{s}G(\tilde{s},t-\tau)|$, where $\tau\le\tau'\le t$ and $\tilde{s}$ is between $s'(t)-s'(\tau)$ and $s(t)-s(\tau)$. Since $|\tilde{s}|\le V_0(t-\tau)$ and $|G|\le C_0(t-\tau)^{-1/2}$

we get the estimate

$$|\Delta G| \leq C_0 V_0 |\varphi(\tau') - \varphi'(\tau')|(t-\tau)^{1/2}. \tag{4.16}$$

Integration with respect to τ yields:

$$\begin{aligned} |W_2 + W_3| &\leq \left| \int_0^t \Delta G \varphi'(\tau) d\tau \right| + \left|\varphi - \varphi'\right|_\sigma \int_0^t C_0 (t-\tau)^{-1/2} d\tau \\ &\leq \left(\frac{2}{3} V_0 C_0 t^{3/2} + \frac{C_0}{2} t^{1/2} \right) \left|\varphi - \varphi'\right|_\sigma := (A_2 t^{3/2} + A_3 t^{1/2}) \left|\varphi - \varphi'\right|_\sigma \end{aligned} \tag{4.17}$$

The estimation for the term $|W_1|$ in (4.15) is a little more involved. Denote $\delta G := G(s-\xi, t) - G(s'-\xi, t)$, then

$$|W_1| = \left| \int \delta G u_0(\xi) d\xi \right| = \left| \int \omega \omega^{-1} \delta G\, u^0 d\xi \right| \underset{\text{Cauchy-Schwarz}}{\leq} \sqrt{I_1}\, ||u_0||_\omega \tag{4.18}$$

where $I_1 = \int |G(s-\xi,t) - G(s'-\xi,t)|^2 e^{-\alpha\xi} d\xi := I_{11} - 2I_{12} + I_{22}$.

Next we deal separately with terms I_{11}, I_{12}, and I_{22}. Denote $a = s(t),\ b = s'(t)$; then by completing the square in the exponent and integrating we get

$$\begin{aligned} I_{12} &= \frac{1}{4\pi} \int t^{-1} \exp\left[-\frac{(s(t)+\xi)^2 + (s'(t)+\xi)^2}{4t} - \alpha\xi \right] d\xi \\ &= \sqrt{2\pi} / \left(4t^{1/2}\right) \exp\left[\frac{\alpha^2 t + \alpha(b+a)}{2} \right] \exp\left[-(\frac{a-b}{2})^2 \right] \end{aligned}$$

The integrals I_{11} and I_{22} are obtained from I_{12} by replacing $a-b$ by 0, and $a+b$ by $2a$ and $2b$ respectively

$$I_{11} = \sqrt{2\pi}/(4t^{1/2}) \exp[t\alpha^2/2 + \alpha a], \quad I_{22} = \sqrt{2\pi}/(4t^{1/2}) \exp[t\alpha^2/2 + \alpha b]$$

Now we rewrite $I_{11} - 2I_{12} + I_{22}$ as a complete square plus a correction

$$\begin{aligned} &I_{11} - 2I_{12} + I_{22} \\ &= \frac{\sqrt{2\pi}}{4\sqrt{t}} \exp\frac{\alpha^2 t}{2} \left\{ \left[\exp\frac{\alpha a}{2} - \exp\frac{\alpha b}{2} \right]^2 + 2\exp\frac{\alpha(b+a)}{2} \left[1 - \exp\left[-\left(\frac{a-b}{2}\right)^2 \right] \right] \right\} \\ &\leq \frac{\sqrt{2\pi}}{4\sqrt{t}} (\frac{a-b}{2})^2 \exp\frac{\alpha^2 t}{2} \left\{ \alpha^2 \exp(\alpha\theta) + 2\exp\frac{\alpha(b+a)}{2} \right\} \end{aligned}$$

Above we used an elementary inequality $|\exp(-x^2) - 1| \leq x^2$.

Upon plugging the just established estimate for I_1 into (4.18) we obtain

$$|W_1| \leq [I_{11} - 2I_{12} + I_{22}]^{1/2} ||u_0||_\omega \leq A_1 \sigma^{3/4} \|u_0\|_\omega \sup_{0<t\leq\sigma} |\varphi - \varphi'| \tag{4.19}$$

As a result we get a contraction for σ sufficiently small:

$$\left|\psi - \psi'\right|_\sigma = \left|K\varphi - K\varphi'\right|_\sigma \le L \sup_{0<t\le\sigma} |W_1 + W_2 + W_3| \le C\sqrt{\sigma}\left|\varphi - \varphi'\right|_\sigma .$$

□

4.3. *Interface temperature*

Results of this section will also be needed in Sec. 10.

PROPOSITION 5. *If the exponent α in the weight satisfies $\alpha \le v_0$ then*

$$u(s(t), t) \le V_0/v_0 + t^{-1/4}\,\|u_0\|_\omega \tag{4.20}$$

Proof. First, the contribution of the source term in (4.12) into the interface temperature is estimated as follows:

$$\left|\int_0^t G(s(t) - s(\tau), t-\tau)v(\tau)d\tau\right| \le \left|\int_0^t \frac{V_0}{2\sqrt{\pi(t-\tau)}}\exp[-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}]d\tau\right|$$
$$\le V_0\left|\int_0^\infty \frac{2}{v_0\sqrt{\pi}}\exp\left(-v_0^2\frac{\eta}{4}\right)d\left(\frac{v_0\sqrt{\eta}}{2}\right)\right| = V_0/v_0.$$

Next we estimate the contribution from the initial data.

$$\int G(s(t)-\xi, t)u_0(\xi)d\xi = \frac{1}{\sqrt{2t\pi}}\int \exp\left(-\frac{(s(t)-\xi)^2}{4t} + \alpha\xi\right)\exp(-\alpha\xi)u_0(\xi)d\xi$$
$$\underset{\text{Cauchy-Schwartz}}{\le} \|u_0\|_\omega \frac{e^{\alpha s + \alpha^2 t}}{\sqrt{2t\pi}}\left(\int \exp(-\frac{[s(t)-\xi+2\alpha t]^2}{2t})d\xi\right)^{1/2}$$
$$= \|u_0\|_\omega t^{-1/4}\exp(\alpha s + \alpha^2 t) \le \|u_0\|_\omega t^{-1/4}\exp(-\alpha v_0 t + \alpha^2 t)$$

The two estimates yield the result of Proposition. □

From now on we shall assume that $\alpha < v_0$.

5. Estimates for the solution in weighted spaces

In the sequel we estimate certain integrals of the error function type:

LEMMA 6. *For $a, b > 0$*

$$\int_a^\infty e^{-b\eta^2}d\eta \le \begin{cases} \exp(-ba^2)/\left(2\sqrt{b}\right), \; \textit{for } a > 1/\sqrt{b} \\ \sqrt{\pi}/\left(2\sqrt{b}\right), \; \textit{for } 0 \le a < 1/\sqrt{b} \end{cases}$$

Proof. If $a\sqrt{b} > 1$ then

$$\int_a^\infty e^{-b\eta^2} d\eta = \frac{1}{\sqrt{b}} \int_{a\sqrt{b}}^\infty e^{-\eta^2} d\eta \leq \frac{1}{\sqrt{b}} \int_{a\sqrt{b}}^\infty \eta e^{-\eta^2} d\eta = \frac{1}{2\sqrt{b}} e^{-ba^2}$$

On the other hand, always for $a > 0$

$$\int_a^\infty e^{-b\eta^2} d\eta \leq \int_0^\infty e^{-b\eta^2} d\eta = \sqrt{\pi}/(2\sqrt{b})$$

□

Also, the following Poincaré type lemma will be instrumental below.

LEMMA 7. *Let* $f \in H^1_\omega$, *then for any* $0 < c \leq 1/\alpha$

$$f^2(0) \leq \left(-\alpha + \frac{1}{c}\right) \|f\|^2_\omega + c \, \|f_x\|^2_\omega \tag{5.21}$$

Proof. Assume first that f is C^1. By integrating from $-\infty$ to 0 we obtain:

$$\begin{aligned} f^2(0) &= \int_{-\infty}^0 (e^{-\alpha x} f^2)_x dx = 2 \int_{-\infty}^0 e^{-\alpha x} f_x f dx - \alpha \int_{-\infty}^0 e^{-\alpha x} f^2 dx \\ &\leq \left(-\alpha + \frac{1}{c}\right) \int e^{-\alpha x} f^2 dx + c \int e^{-\alpha x} f_x^2 dx \end{aligned}$$

For $c = 1/\alpha$ one obtains an inequality, similar to the ordinary Poincaré's inequality. Thus the role of $1/\alpha$ is similar to the interval length for the Poincaré lemma on a finite interval.

□

5.1. *Energy estimates*

The energy estimates below are performed in the interface-attached coordinate frame (3.8); we abuse notation and still use x instead of ζ in (3.8).

THEOREM 8. *Denote* $\kappa = \alpha(v_0 - \alpha)$, *where* $0 < \alpha < v_0$, *then*

$$\|u(.,t)\|^2_\omega \leq C + \|u_0\|^2_\omega \exp(-\kappa t), \quad \int_0^t \|u_x\|^2_\omega dt \leq C_0 t + \|u_0\|^2_\omega \tag{5.22}$$

where $C > 0$ *depends on* α *and* ***sublinear*** *kinetics* g.

Proof. We multiply (3.8) by $ue^{-\alpha x}$ and integrate by parts separately on the intervals $(-\infty, 0)$ and $(0, \infty)$ taking into account the interface conditions,

$$\frac{1}{2}\frac{d}{dt} \|u\|^2_\omega = \int e^{-\alpha x} u(u_{xx} + vu_x) dx = \frac{\alpha(v+\alpha)}{2} \|u\|^2_\omega - \|u_x\|^2_\omega - u(0,t)v \tag{5.23}$$

The (positive) term $-u(0,t)v$ is due to the jump condition (3.10). By sublinearity (see Theorem 1) $|v/u| = |g(u)/u| \to 0$ as $u \to \infty$. Therefore if $|u| > M$, then $|v| < \varepsilon|u|$ for an appropriate M depending on ε. On the other hand, for $|u| \le M$, $g(u) \le g(M)$ (by monotonicity of g). As a result for the last term we obtain

$$\begin{aligned} |u(0,t)|\,|v(t)| = |u(0,t)|g(u(0,t)) &\le g(M)M + \varepsilon|u|^2 := K + \varepsilon|u|^2 \\ &\underset{(5.21)}{\le} K + \varepsilon(1/c - \alpha)\,\|u\|^2_\omega + \varepsilon c\,\|u_x\|^2_\omega \end{aligned} \tag{5.24}$$

Substitution into (5.23) yields

$$\frac{1}{2}\frac{d}{dt}\,\|u\|^2_\omega \le K + \frac{1}{2}\left[\alpha(v+\alpha) + 2(\frac{1}{c} - \alpha)\varepsilon\right]\|u\|^2_\omega - (1-\varepsilon c)\,\|u_x\|^2_\omega \tag{5.25}$$

To optimize the estimate we set $c = 1/\alpha$, $\varepsilon = \alpha$ which results in

$$\frac{d}{dt}\,\|u\|^2_\omega \le 2K - \kappa\,\|u\|^2_\omega \tag{5.26}$$

with $\kappa = \alpha(v_0 - \alpha)$. By Gronwall's Lemma it immediately yields

$$\|u\|^2_\omega \le \|u_0\|^2_\omega \exp(-\kappa t) + 2K/\kappa$$

For $\varepsilon = \alpha/2$, $c = 1/\alpha$ (5.25) can be rearranged as

$$\|u_x\|^2_\omega \le 2K - \kappa\,\|u\|^2_\omega - \frac{d}{dt}\,\|u\|^2_\omega$$

Integrating it we get the integral estimate for $\|u_x\|^2_\omega$:

$$0 < \int_0^t \|u_x\|^2_\omega\,dt \le 2Kt - \kappa\int_0^t \|u\|^2_\omega\,dt - \|u\|^2_\omega + \|u_0\|^2_\omega \le 2Kt + \|u_0\|^2_\omega\,.$$

□

COROLLARY 9. *The contribution from the initial conditions decays exponentially:*

$$\|u_2(.,t)\|^2_\omega \le \|u_0\|^2_\omega \exp(-\kappa t) \tag{5.27}$$

Proof. The contribution from the initial conditions u_2 is a smooth solution (no jump at zero) of the Cauchy problem for the heat equation in (3.8). For u_2 the calculation similar to the one in (5.23) leads to (5.23) with the jump term $-u(0,t)v$ missing. Thereafter, a simplified version of the proof leads to the estimate (5.27). □

By combining the estimates on the total solution and on the contribution from the initial conditions we estimate the contribution from the interface

COROLLARY 10.

$$\|u_1(.,t)\|_\omega^2 \leq 4K/\kappa + 2\,\|u_0\|_\omega^2 \exp(-\kappa t) \tag{5.28}$$

Proof. $\|u_1(.,t)\|_\omega^2 = \|u(.,t) - u_2(.,t)\|_\omega^2 \underset{\text{triangle}}{\leq} 2\,\|u(.,t)\|_\omega^2 + 2\,\|u_2(.,t)\|_\omega^2$ ☐

The estimate of Theorem 8 can be made more explicit if the *kinetics function is bounded* instead of sublinear.

COROLLARY 11. *For the bounded kinetics*

$$\|u(.,t)\|_\omega^2 \leq V_0^2/\,(2\alpha\kappa) + \|u_0\|_\omega^2 \exp{(-\kappa t)} \tag{5.29}$$

$$\|u_1(.,t)\|_\omega^2 \leq V_0^2/\,(\alpha\kappa) + 2\,\|u_0\|_\omega^2 \exp(-\kappa t) \tag{5.30}$$

Proof. The estimate in (5.24) simplifies as follows:

$$|u|\,|v| \leq |u|V_0 \leq V_0^2/(4\alpha) + \alpha|u|^2$$

i.e., the constant K is replaced by $V_0^2/(4\alpha)$. The subsequent estimates change accordingly and lead to the results in (5.29)–(5.30). ☐

REMARK 12. If we choose $\alpha = \lambda v_0$, then for large t the bound becomes

$$\|u\|_\omega^2 \leq \frac{1}{2\lambda(1-\lambda)}\frac{V_0^2}{v_0^3} + \varepsilon.$$

This provides an estimate on the diameter of the absorbing ball in Sec. 8.

5.2. *Spatial decay of the interface contribution*

For any fixed t we show that norms of the ”tails” (for $|\zeta|$ large) of the interface contribution to the solution tend to zero, uniformly for any bounded set of interface velocities. This is necessary for H_ω-compactness of the absorbing set.

PROPOSITION 13. *For any $M > 1$ the norms of the restrictions of $u_1(t,.)$ to the intervals $(-\infty,-M)$ and (M,∞) decay exponentially in M uniformly in time:*

$$\|u_1(t,.)\|_{-M} := \left(\int_{-\infty}^{-M} \omega(\zeta)\left[\int_0^t G(\zeta + s(t) - s(\tau), t-\tau)v(\tau)d\tau\right]^2 d\zeta\right)^{1/2}$$

$$\le c_- \exp\left(-\frac{v_0-\alpha}{2}M\right) \tag{5.31}$$

$$\begin{aligned}\|u_1(t,.)\|_M &:= \left(\int_M^\infty \omega(\zeta)\left[\int_0^t G(\zeta+s(t)-s(\tau),t-\tau)v(\tau)d\tau\right]^2 d\zeta\right)^{1/2}\\ &\le c_+\exp\left(-\frac{\alpha}{2}M\right) \end{aligned} \tag{5.32}$$

where $c_-, c_+ > 0$ *depend only on the kinetics and the weight.*

Proof. For brevity we write $s = s(t),\ s' = s(\tau),\ \Delta = s(t) - s(\tau),\ \tilde{\tau} = t - \tau$. Note that by the "triangle" inequality,

$$\left\|\int_0^t G(.+\Delta, t-\tau)v(\tau)d\tau\right\|_{\pm M} \le \sup_\tau |v(\tau)| \int_0^t \|G(.+\Delta,t-\tau)\|_{\pm M}\, d\tau \tag{5.33}$$

Next we evaluate the norm under the integral sign. We start with the evaluation ahead of the interface, $\zeta < -M$.

$$\begin{aligned}\int_{-\infty}^{-M} \omega(\zeta)\, G^2(\zeta+\Delta,\tilde{\tau})d\zeta &= \frac{1}{4\pi\tilde{\tau}}\int_{-\infty}^{-M}\exp\left[-\frac{(\zeta+\Delta)^2}{2(t-\tau)}-\alpha\zeta\right]d\zeta\\ &= \frac{1}{4\pi\tilde{\tau}}\exp\left(-\frac{\Delta^2-[\Delta+\tilde{\tau}\alpha]^2}{2\tilde{\tau}}\right)\int_{-\infty}^{-M}\exp\left[-\frac{\{\zeta+(\Delta+\tilde{\tau}\alpha)\}^2}{2\tilde{\tau}}\right]d\zeta\\ &\le \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\left[\alpha\left\{\Delta+\tilde{\tau}\frac{\alpha}{2}\right\}\right]\exp\left[-\frac{\{-M+(\Delta+\tilde{\tau}\alpha)\}^2}{2\tilde{\tau}}\right]\end{aligned}$$

To obtain the last inequality we completed the square in the exponent and employed Lemma 6. (recall that $M > 1$, and $\alpha - v_0 < 0$). Furthermore, since $\Delta < -v_0\tilde{\tau}$ and $(-M+\tilde{\tau}(\alpha-v_0))^2 \ge 2M\tilde{\tau}(\alpha-v_0)$ we continue as follows

$$\begin{aligned}&\le \exp\left[\alpha\left(-v_0+\frac{\alpha}{2}\right)\tilde{\tau}\right]\frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\left[-\frac{\{-M+\tilde{\tau}(\alpha-v_0)\}^2}{2\tilde{\tau}}\right]\\ &\le \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\left[\alpha\left(-v_0+\frac{\alpha}{2}\right)\tilde{\tau}\right]\exp[-M(v_0-\alpha)]\end{aligned}$$

Then integration with respect to τ yields

$$\begin{aligned}\|u_1(t,.)\|_{-M} &\underset{(5.33)}{\le} \sup_\tau|v(\tau)|\int_0^t\{\int_{-\infty}^{-M}\omega(\zeta)|G|^2d\zeta\}^{1/2}d\tilde{\tau}\\ &\le e^{-M(v_0-\alpha)}\sup_\tau|v(\tau)|\int_0^t\left\{\frac{1}{\sqrt{2\pi\tilde{\tau}}}\exp\left[-\alpha\left(v_0-\frac{\alpha}{2}\right)\tilde{\tau}\right]\right\}^{1/2}d\tilde{\tau}\\ &\le cV_0e^{-M(v_0-\alpha)/2}[\alpha(v_0-\alpha/2)]^{-3/4}\end{aligned}$$

For the tail *behind the interface* $\zeta > M$ we have

$$\begin{aligned}&\frac{1}{4\pi\tilde{\tau}}\int_M^\infty \exp\left[-\frac{(\zeta+\Delta)^2}{2\tilde{\tau}}-\alpha\zeta\right]d\zeta\\ &\underset{\zeta>M}{\le} \exp\left[-\frac{\alpha}{2}M\right]\frac{1}{4\pi}\int \tilde{\tau}^{-1}\exp\left[-\frac{(\zeta+\Delta)^2}{2\tilde{\tau}}-\frac{\alpha\zeta}{2}\right]d\zeta\\ &\le \exp\left[-\frac{\alpha}{2}M\right]\frac{c}{\sqrt{\tilde{\tau}}}\exp\left(-\frac{\alpha}{2}\left(v_0-\frac{\alpha}{4}\right)\tilde{\tau}\right)\end{aligned}$$

Similarly to the $\zeta < -M$ case, integration with respect to τ yields

$$\|u_1(t,.)\|_M \le cV_0e^{-M\alpha/2}\left[\frac{\alpha}{2}(v_0-\frac{\alpha}{4})\right]^{-3/4}$$

□

6. Spatial derivative of the solution

Estimates below are performed in the *"laboratory" system of coordinates* (x,t).

THEOREM 14. *The* H_ω*-norm of the spatial derivative of the solution satisfies*

$$\|u_x(.,t)\|_\omega^2 \le \left(\frac{e^{-\alpha v_0 t}}{\sqrt{t\pi}}+\alpha\sqrt{2}e^{-\alpha(v_0-\alpha)t}\right)\|u_0\|_\omega^2+(CV_0)^2\left[\frac{\alpha}{2}\left(v_0-\frac{\alpha}{2}\right)\right]^{-3/2} \tag{6.34}$$

The principal application of the theorem is the following obvious

COROLLARY 15. *For large* t, *uniformly in any ball* $\|u_0\|_\omega \le R$,

$$\|u_x(.,t)\|_\omega^2 \le (CV_0)^2\left[\frac{\alpha}{2}\left(v_0-\frac{\alpha}{2}\right)\right]^{-3/2}+\varepsilon := C_{ux}^2+\varepsilon \tag{6.35}$$

Similarly to the proof for the solution itself, the proof of (6.34) is split into two parts:

6.1. *Contribution from initial data*

PROPOSITION 16. *The derivative of the contribution from initial data satisfies*

$$\left\|(u_2)_x(.,t)\right\|_\omega^2 \le \left(\exp(-\alpha v_0 t)/\sqrt{t\pi}+\alpha\sqrt{2}\exp\{-\alpha(v_0-\alpha)t\}\right)\|u_0\|_\omega^2 \tag{6.36}$$

We note that the proof below is based on a curious trick that is similar to the well-known way of computing the Gaussian integral.

Proof. For $(u_2)_x(.,t) = (T_2(t)u_0)_x$, we have:

$$
\begin{aligned}
\left\|(u_2)_x\,(.,t)\right\|_\omega^2 &= \int \left|\int G_x(x-\xi,t)u_0(\xi)d\xi\right|^2 \omega(x-s(t))dx \\
&= \int \left(\int G_x(x-\xi,t)u_0(\xi)d\xi \int G_x(x-\eta,t)u_0(\eta)d\eta\right)\omega(x-s)dx \\
&= \int \left(\int_{\mathbb{R}^2} G_x(x-\xi,t)G_x(x-\eta,t)u_0(\xi)u_0(\eta)d\eta d\xi\right)\omega(x-s)dx \\
&\le \int \left(\int_{\mathbb{R}^2} |G_x(x-\xi,t)|\,|G_x(x-\eta,t)|\frac{u_0(\xi)^2+u_0(\eta)^2}{2}d\eta d\xi\right)\omega(x-s)dx
\end{aligned}
$$

We replace the interior double integral by the two repeated ones, choosing a convenient order of integration for each one:

$$
\begin{aligned}
&\frac{1}{2}\int \omega(\xi)u_0(\xi)^2 \int \left(\int |G_x(x-\eta,t))|d\eta\right)\omega^{-1}(\xi)|G_x(x-\xi,t)|\omega(x-s)dxd\xi \\
&+\frac{1}{2}\int \omega(\eta)u_0(\eta)^2 \int \left(\int |G_x(x-\xi,t))|d\xi\right)\omega^{-1}(\eta)|G_x(x-\eta,t)|\omega(x-s)dxd\eta
\end{aligned}
$$

Since

$$
\int |G_x(x-\eta,t)|d\eta = \frac{1}{2\sqrt{t\pi}}\int \left|\frac{x-\eta}{2t}\right|\exp\left(-\frac{[(x-\eta)]^2}{4t}\right)d\eta = \frac{1}{\sqrt{t\pi}} \tag{6.37}
$$

for the interior integrals we get

$$
\begin{aligned}
&\int \left(\int |G_x(x-\eta,t)d\eta\right)\omega^{-1}(\xi)|G_x(x-\xi,t)|\omega(x-s)dx \\
&\underset{(6.37)}{=} \frac{1}{\sqrt{t\pi}}\int \omega^{-1}(\xi)|G_x(x-\xi,t)|\omega(x-s)dx \\
&= \frac{1}{\sqrt{t\pi}}\int \frac{|x-\xi|}{2t}\exp\left(-\frac{(x-\xi)^2}{4t}-\alpha(x-s)+\alpha\xi\right)dx
\end{aligned}
$$

Next we split the domain of integration into two parts $x > \xi$ and $x < \xi$:

$$
\begin{aligned}
&\exp(\alpha s+\alpha^2 t)\int_\xi^\infty \frac{1}{\sqrt{\pi t}}\left(\frac{x-\xi+2\alpha t}{2t}-\alpha\right)\exp\left(-\frac{[x-\xi+2\alpha t]^2}{4t}\right)dx \\
&+\exp(\alpha s+\alpha^2 t)\int_{-\infty}^\xi \frac{1}{\sqrt{\pi t}}\left(\frac{\xi-x-2\alpha t}{2t}+\alpha\right)\exp\left(-\frac{[x-\xi+2\alpha t]^2}{4t}\right)dx \\
&\underset{y=[x-\xi+2\alpha t]/(2\sqrt{t})}{=} 2e^{\alpha s+\alpha^2 t}\frac{1}{\sqrt{\pi t}}\left(\int_{\alpha\sqrt{t}}^\infty ye^{-y^2}dy+\alpha\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{\alpha\sqrt{t}}\alpha e^{-y^2}dy\right) \\
&\le 2\exp(\alpha s+\alpha^2 t)\left[\frac{\exp(-\alpha^2 t)}{2\sqrt{\pi t}}+\frac{\alpha}{\sqrt{2}}\right]
\end{aligned}
$$

Since $s(t) < -v_0 t$ it yields,

$$\left\|(u_2)_x\,(.,t)\right\|_\omega^2 \leq \exp(-\alpha v_0 t + \alpha^2 t)\left(\frac{\exp(-\alpha^2 t)}{\sqrt{\pi t}} + \sqrt{2}\alpha\right)\|u_0\|_\omega^2 \tag{6.38}$$

□

6.2. *Contribution from the interface*

PROPOSITION 17. *The H_ω-norm of the derivative of the contribution from the free interface is uniformly bounded for all time (provided that $\alpha < 2v_0$)*

$$\|(u_1)_x(.,t)\|_\omega \leq CV_0[(v_0 - \alpha/2)\alpha/2]^{-3/4} \tag{6.39}$$

Proof. From the "triangle" inequality, it is obvious that

$$\|(u_1)_x(t)\|_\omega \leq \int_0^t \|G_x(.-s(\tau), t-\tau)\|_\omega\,|v|d\tau \leq V_0 \int_0^t \|G_x(.-s(\tau), t-\tau)\|_\omega\,d\tau$$

Introduce again $\tilde{\tau} = t - \tau$. To estimate the norm in the integrand

$$\|G_x\|_\omega^2 = \frac{1}{4\pi}\int (x - s(\tau))^2/(4\tilde{\tau}^3)\exp[-(x - s(\tau))^2/\,(2\tilde{\tau}) - \alpha(x - s(t))]dx,$$

we complete the square in the exponent to $\{(x - s(\tau) + \tilde{\tau}\alpha)\}^2/2\tilde{\tau}$, and then directly evaluate the Gaussian integrals:

$$\begin{aligned}\|G_x\|_\omega^2 &= (c_1\tilde{\tau}^{-3/2} + c_2\tilde{\tau}^{-1} + c_3\tilde{\tau}^{-1/2})\exp\{\alpha[s(t) - s(\tau) + \tilde{\tau}\alpha/2]\}\\ &\leq \tilde{\tau}^{-3/2}(c_1 + \tilde{\tau}^{1/2}c_2 + \tilde{\tau}c_3)\exp\left(-\alpha(v_0 - \alpha/2)\tilde{\tau}\right)\end{aligned}$$

From this, by using convexity of the square root, we obtain

$$\begin{aligned}&\|(u_1)_x(.,t)\|_\omega \leq V_0\int_0^t (\|G_x\|_\omega^2)^{1/2}d\tau\\ &\leq V_0\int_0^t (c_1^{1/2}\tilde{\tau}^{-3/4} + c_2^{1/2}\tilde{\tau}^{-1/2} + c_3^{1/2}\tilde{\tau}^{-1/4})\exp\left(-\frac{\alpha}{2}\left(v_0 - \frac{\alpha}{2}\right)\tilde{\tau}\right)d\tilde{\tau}\\ &\leq V_0(c_1^{1/2}\Gamma(1/4)a^{-1/4} + c_2^{1/2}\Gamma(1/2)a^{-1/2} + c_3^{1/2}\Gamma(3/4)a^{-3/4}) \leq CV_0 a^{-3/4}\end{aligned}$$

where $a = (v_0 - \alpha/2)\alpha/2$, and C is an absolute constant. The integrals from 0 to t above were estimated by the integrals from 0 to ∞. □

7. Dependence of solutions on initial data

Next *we establish continuity of the evolution operator* by proving that solutions of FIP (3.8)–(3.10) depend continuously on the initial data. This will be used in the sequel to demonstrate spatial smoothness of the elements of the attractor.

THEOREM 18. *Solutions of (3.8)–(3.10) depend on initial conditions continuously in* H_ω*: Let* $\{u(x,t), s(t)\}$, $\{u'(x,t), s'(t)\}$ *be solutions with initial data* $u_0, u_0' \in B_R \subset H_\omega$*, where* $B_R = \{w : \|w\|_\omega \leq R\}$*. Then there exists* $\sigma > 0$ *that depends on R alone so that for* $0 < t < \sigma$

$$\sup_{0<t<\sigma} |v(t) - v'(t)| < c\sigma^{-1/4} \left\|u_0 - u_0'\right\|_\omega , \tag{7.40}$$

$$\left\|u - u'\right\|_\omega < c \left\|u_0 - u_0'\right\|_\omega \tag{7.41}$$

Furthermore, uniformly on the ball B_R*, with C depending solely on* ω*:*

$$\int_0^t \left\|u_x(.,\tau) - u_x'(,\tau)\right\|_\omega^2 d\tau \leq e^{C\sqrt{t}} \left\|u_0 - u_0'\right\|_\omega^2$$

REMARK 19. The argument extending the Theorem to any fixed time is based on the a priori estimates and follows closely the proof of global existence.

REMARK 20. If the initial data are in H_ω^1 then a similar argument yields the following estimate, analogous to (7.41), thus guaranteeing continuous dependence on the initial data in H_ω^1:

$$\left\|u(t) - u'(t)\right\|_{\omega,1} \leq C \ \left\|u_0 - u_0'\right\|_{\omega,1} \tag{7.42}$$

The proof of Theorem 18 consists of two parts. First we establish continuity of the interface velocity (7.40) and then use it for the energy estimate (7.41).

7.1. *Estimates for* $v - v'$

The estimates of this section are very similar to those in Sec. 4.2; they are performed in *the laboratory coordinate system.* Introduce the norm

$$\left[\left|v - v'\right|\right]_{\sigma,1/4} = \sup_{0<\tau<\sigma} \tau^{1/4}|v(\tau) - v'(\tau)| \tag{7.43}$$

Here the exponent $1/4$ is selected for simplicity of presentation; it is sufficient to take $\tau^{1/2-\varepsilon}$ for any $0 < \varepsilon < 1/2$

Let v and v' be solutions of (4.11) with initial data u_0 and u_0', then

$$\begin{aligned}
|v - v'| &= |g\left(\int G(s(t) - \xi, t)u_0(\xi)d\xi - \int_0^t G(s(t) - s(\tau), t - \tau)v(\tau)d\tau\right) \\
&\quad -g\left(\int G(s'(t) - \xi, t)u_0'(\xi)d\xi - \int_0^t G(s'(t) - s'(\tau), t - \tau)v'(\tau)d\tau\right)| \\
&\le L\left|\int G(s(t) - \xi, t)u_0(\xi)d\xi - \int G(s'(t) - \xi, t)u_0'(\xi)d\xi\right| \\
&\quad +L\left|\int_0^t G(s(t) - s(\tau), t - \tau)v(\tau)d\tau - \int_0^t G(s'(t) - s'(\tau), t - \tau)v'(\tau)d\tau\right|,
\end{aligned}$$

where L is the Lipschitz constant of g. Next we employ a "coordinate descent":

$$\begin{aligned}
&\le L\left|\int (G(s(t) - \xi, t) - G(s'(t) - \xi, t))u_0(\xi)d\xi\right| \\
&\quad + L\left|\int G(s'(t) - \xi, t)(u_0 - u_0')d\xi\right| + L\left|\int_0^t G(s(t) - s(\tau), t - \tau)(v - v')d\tau\right| \\
&\quad + L\left|\int_0^t [G(s(t) - s(\tau), t - \tau) - G(s'(t) - s'(\tau), t - \tau)]v'd\tau\right| \\
&:= L(W_0 + W_1 + W_2 + W_3)
\end{aligned}$$

The term W_0 is estimated as follows,

$$|W_0(s', t)| \underset{\text{Cauchy-Schwartz}}{\le} ||u_0 - u_0'||_\omega \left|\int \omega^{-1}(\xi)|G(s', t, \xi, 0)|^2 d\xi\right|^{1/2}$$

Since $\alpha s' + \alpha^2 t/2 \le -\alpha v_0 t + \alpha^2 t/2 < 0$ for $\alpha < 2v_0$, the last integral is:

$$\int \omega^{-1}|G(s' - \xi, t)|^2 d\xi = \frac{\exp(\alpha s' + \alpha^2 t/2)}{2\pi t}\int \exp(-\frac{[(s' - \xi) - \alpha t]^2}{2t})d\xi \le c/\sqrt{t}$$

Thus, $|W_0| \le A_0 t^{-1/4}||u_0 - u_0'||_\omega$. The term W_1 is identical to W_1 in (4.18) and therefore, by (4.19) $|W_1| \le |v(t) - v'(t)|A_1 t^{1/2}||u_0||_\omega$. For the estimate of W_2, after a simple algebra we replace the exponential by 1 and integrate to obtain:

$$\begin{aligned}
W_2(t) &= \left|\int_0^t \tau^{-1/4}(4\pi(t - \tau))^{-1/2}\exp\left\{-\frac{[s(t) - s(\tau)]^2}{4(t - \tau)}\right\}\tau^{1/4}(v(\tau) - v'(\tau))d\tau\right| \\
&\le A_2 t^{1/4}\left[|v - v'|\right]_{\sigma,1/4}
\end{aligned}$$

Finally we estimate $W_3 = \left|\int_0^t \Delta G v'(\tau)d\tau\right|$. For $\Delta G = G(s(t) - s(\tau), t - \tau) - G(s'(t) - s'(\tau), t - \tau)$ we have

$$\begin{aligned}
|\Delta G| &\underset{(4.16)}{\le} C_0 V_0|v(\tau') - v'(\tau')|(t - \tau)^{1/2} \\
&\underset{\tau\le\tau'\le t}{\le} C_0 V_0|v - v'|(\tau')^{1/4}(t - \tau)^{1/2}\tau^{-1/4} \le C_0 V_0\left[|v - v'|\right]_{\sigma,1/4}(t - \tau)^{1/2}\tau^{-1/4}
\end{aligned}$$

Integration with respect to τ yields:

$$W_3(t) \leq \left[|v - v'|\right]_{\sigma,1/4} C_0 V_0^2 \int_0^t \tau^{-1/4}(t-\tau)^{1/2} d\tau = A_3 t \left[|v - v'|\right]_{\sigma,1/4}$$

We collect the estimates for W_0 through W_3 to obtain

$$\begin{aligned}&\left|v(t) - v'(t)\right| \leq LA_0 t^{-1/4} ||u_0 - u_0'||_\omega \\ &\quad + L|v(t) - v'(t)| A_1 t^{1/2} ||u_0||_\omega + LA_2 t^{1/4} \left[|v - v'|\right]_{\sigma,1/4} + LA_3 t \left[|v - v'|\right]_{\sigma,1/4}\end{aligned}$$

Upon multiplying this inequality by $t^{1/4}$ we get:

$$\begin{aligned}&\left[|v - v'|\right]_{\sigma,1/4} \leq LA_0 ||u_0 - u_0'||_\omega + LA_1 \left[|v - v'|\right]_{\sigma,1/4} \sigma^{1/2} ||u_0||_\omega \\ &\quad + LA_2 \sigma^{1/2} \left[|v - v'|\right]_{\sigma,1/4} + LA_3 \sigma^{5/4} \left[|v - v'|\right]_{\sigma,1/4} \\ &\leq LA_0 ||u_0 - u_0'||_\omega + L\{A_1 \sigma^{1/2} ||u_0||_\omega + A_2 \sigma^{1/2} + A_3 \sigma^{5/4}\} \left[|v - v'|\right]_{\sigma,1/4}\end{aligned}$$

If σ is sufficiently small then the last inequality yields

$$\left[|v - v'|\right]_{\sigma,1/4} \leq C \|u_0 - u_0'\|_\omega \tag{7.44}$$

The choice of σ depends only on $\|u_0\|_\omega$; therefore (7.44) can be interpreted as uniformly continuous dependence of v on the initial data in any ball $\|u_0\|_\omega \leq R$.

7.2. *Energy estimates for $u - u'$*

In this subsection we once again abuse notation and denote by $u(x, t)$ and $u'(x, t)$ the two solutions of the free boundary problem in *the frame attached to the free boundary* (3.8)–(3.10) that correspond to the solutions in the laboratory coordinate system of the previous subsection. Thus u and u' solve the problems

$$\begin{aligned}&u_t = u_{xx} + [u_x(0, t)]u_x, \quad g(u(0, t)) = [u_x(0, t)], \quad u(x, 0) = u_0(x), \\ &u_t' = u_{xx}' + [u_x'(0, t)]u_x', \quad g(u'(0, t)) = [u_x'(0, t)], \quad u'(x, 0) = u_0'(x).\end{aligned}$$

The difference $w = u - u'$ solves the following problem

$$\begin{aligned}&w_t = w_{xx} + [u_x(0, t)]w_x + [w_x(0, t)]u_x'; \quad w|_{t=0} = u_0 - u_0', \\ &[w_x(0, t)] = (g(\theta))' w(0, t) = g(u(0, t)) - g(u'(0, t)),\end{aligned} \tag{7.45}$$

where θ is an intermediate value between $u(0, t)$ and $u'(0, t)$, We multiply the equation by ωw and integrate to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_\omega^2 &= \int \omega w_{xx} w dx + [u_x(0,t)]\int \omega w_x w dx + [w_x(0,t)]\int \omega u_x' w dx\\ &= -[w_x]w|_0 - \|w_x\|_\omega^2 + \int \alpha\omega w_x w dx\\ &\quad + [u_x(0,t)]\int \omega w_x w dx + [w_x(0,t)]\int \omega u_x' w dx\end{aligned}$$

To estimate the first term above, we use Lemma 7.

$$|[w_x(0,t)]w(0,t)| \underset{(7.45)}{\leq} Lw(0,t)^2 \leq L\left(\frac{1}{\varepsilon}\|w\|_\omega^2 + \varepsilon\|w_x\|_\omega^2 - \alpha\|w\|_\omega^2\right) \tag{7.46}$$

(recall that $-(g(\theta))' \leq L$). Next we note that

$$\left|\int \omega w_x w dx\right| \leq \frac{1}{2}(\frac{1}{\varepsilon}\|w\|_\omega^2 + \varepsilon\|w_x\|_\omega^2)$$

Therefore

$$|[u_x(0,t)]\int \omega w_x w dx| \leq \frac{V_0}{2}(\frac{1}{\varepsilon}\|w\|_\omega^2 + \varepsilon\|w_x\|_\omega^2) \tag{7.47}$$

Also, as follows from Theorem 14, $\left\|u_x'(.,t)\right\|_\omega^2 \leq C_0^2/\sqrt{t}$ for $t \leq 1$. We use Cauchy-Schwartz inequality to obtain

$$\begin{aligned}\left|[w_x(0,t)]\int \omega u_x' w dx\right| &\leq L|w(0,t)|\ \left\|u_x'\right\|_\omega\ \|w\|_\omega\\ &\leq (L/2)\,C_0 t^{-1/4}(\|w\|_\omega^2 + |w(0,t)|^2)\\ &\leq Ct^{-1/4}(\|w\|_\omega^2 + \|w\|_\omega^2/\varepsilon + \varepsilon\|w_x t\|_\omega^2 - \alpha\|w\|_\omega^2)\end{aligned}$$

Collecting the estimates for different terms we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_\omega^2 &\leq [-1 + \varepsilon(L + V_0/2 + \alpha/2 + Ct^{-1/4})]\,\|w_x\|_\omega^2\\ &\quad + \Big[L\,(1/\varepsilon - \alpha) + \alpha + (V_0 + \alpha)\,/\,(2\varepsilon) + Ct^{-1/4}(1 + 1/\varepsilon - \alpha)\Big]\,\|w\|_\omega^2\\ &\leq -\|w_x\|_\omega^2/2 + ct^{-1/2}\,\|w\|_\omega^2/2\end{aligned}$$

where on the last step we have chosen $\varepsilon = \left[2(L + V_0/2 + \alpha/2) + 2Ct^{-1/4}\right]^{-1}$ so that the coefficient of $\|w_x\|_\omega^2$ is $-1/2$. Then the coefficient of $\|w\|_\omega^2$ becomes $c_0 + c_1 t^{-1/4} + c_2 t^{-1/2}$ which is $\leq t^{-1/2}/2$ for $t < \sigma$. Thus we obtain

$$\frac{d}{dt}\|w\|_\omega^2 + \|w_x\|_\omega^2 \leq ct^{-1/2}\,\|w\|_\omega^2. \tag{7.48}$$

From this inequality we get first that $\frac{d}{dt}\|w\|_\omega^2 \le ct^{-1/2}\|w\|_\omega^2$ and therefore by the Gronwall lemma

$$\|w\|_\omega^2 \le \|w_0\|_\omega^2 \exp(2ct^{1/2});$$

this yields the first estimate in the statement of Theorem 18. At the same time by rearranging (7.48) and integrating with respect to t, we obtain

$$\begin{aligned}\int_0^t \|w_x\|_\omega^2 \, d\tau &\le \int_0^t (c\tau^{-1/2}\|w\|_\omega^2 - \frac{d}{dt}\|w\|_\omega^2) d\tau \\ &= \|w_0\|_\omega^2 \exp(2ct^{1/2}) - \|w\|_\omega^2 \le 2\|w_0\|_\omega^2 \exp(2ct^{1/2})\end{aligned} \tag{7.49}$$

which is the contents of the second statement of Theorem 18. □

REMARK 21. For $t \ge \sigma > 0$, the last estimate could be modified as follows

$$\int_0^t \|w_x\|_\omega^2 \, d\tau \le c_1 \exp(ct)\|w_0\|_\omega^2$$

8. Absorbing set and attractor

In the current section we establish existence of a bounded absorbing set and of the attractor, which is compact in H_ω. We rephrase Corollary 9 and the estimate in (5.30) in the following form:

PROPOSITION 22. (i) T_2 *is uniformly exponentially contracting in* H_ω*:*

$$\|T_2(t)u_0\|_\omega \le C\exp(-\kappa t/2)\|u_0\|_\omega,$$

for $t \ge t_0$ *and any* $t_0 > 0$ *(*C *depends on* t_0 *for small* t_0; *this dependence can be ignored for* $t_0 > 1$*), where* $\kappa = \alpha(v_0 - \alpha) > 0$*;*

(ii) *For any* $\varepsilon > 0$, *the following ball is a global absorbing set in* H_ω*:*

$$B_{abs} := \{u \in H_\omega : \ \|u\|_\omega \le V_0/\sqrt{\alpha\kappa} + \varepsilon\}$$

Note that the radius of the absorbing ball reflects the contribution of the free interface alone. Next we prove that the free-interface contribution to the evolution, i.e. the family of operators $T_1(t)$ is *uniformly compact*:

PROPOSITION 23. *For any* $t_0 > 0$ *the orbit* $\Omega(t_0) = \cup_{t \ge t_0} T_1(t)B_r$ *of any ball* $B_r = \{u_0 : \|u_0\|_\omega \le r\}$ *is relatively compact in* H_ω.

Proof. We construct a finite ε-net for the orbit $\Omega(t_0)$. First we employ Proposition 13. and select M so that

$$\int_{|\zeta|>M} |(T_1(t)u_0)(\zeta,t)|^2\,\omega(\zeta)d\zeta = \int_{|\zeta|>M}\left(\int_0^t Gv(\tau)d\tau\right)^2 \omega d\zeta \le \varepsilon^2 \tag{8.50}$$

for all $t > t_0$. We have proved that $\Omega(t_0)$ is a bounded set in H^1_ω-norm (see (5.30) and Proposition 17), then according to Rellich's theorem on the finite interval $[-M, M]$ the restriction of $\Omega(t_0)$ is imbedded compactly in the corresponding restriction $H_\omega([-M, M])$ (note that on the finite interval the weight is irrelevant, for example, ω can be taken $\omega \equiv 1$) Therefore a finite ε-net in $H_\omega([-M, M])$ can be constructed. The estimate in (8.50) shows that for all elements of $\Omega(t_0)$ the share of the norm that is responsible for the exterior of $[-M, M]$ is small; therefore by extending elements of the ε-net as 0 beyond this interval we obtain an ε-net in $H_\omega(-\infty, \infty)$. □

Now we can apply the general result [19, Chap. I] that in our situation can be stated as follows:

THEOREM 24. *The ω-limit set of the absorbing ball B_{abs}, $\mathcal{A} = \cap_{t_0\ge 0}\overline{\Omega(t_0)}$, is a global compact attractor for the space H_ω; $\mathcal{A}$ is the maximal attractor in H_ω and it is connected.*

Since, by definition the attractor is both positive and negative invariant with respect to the evolution, the mapping $T(t)$ is one-to-one on the attractor. As an application we obtain the following important characterization of the attractor:

COROLLARY 25. *The attractor $\mathcal{A}$ consists of H^1_ω functions that satisfy*

$$\phi(0,t) \le V_0/v_0, \quad \|\phi\|_\omega \le V_0/\sqrt{2\alpha\kappa}, \tag{8.51}$$

$$\|\phi_x\|_\omega \le B(\alpha) := CV_0[(v_0 - \alpha/2)\alpha/2]^{-3/4} \tag{8.52}$$

where C is an absolute constant (a combination of some Γ-functions).

Proof. Since the mapping is onto, given $\phi \in \mathcal{A}$ for any t there exist $\psi \in \mathcal{A}$, so that $\phi = T(t)\psi$. For $t \to \infty$ the estimate (4.20) yields the estimate on the interface temperature (8.51). Similarly, by taking into account exponential decay of the contribution from initial data as $t \to \infty$, we obtain the norm estimates from (5.29) and (6.34). □

REMARK 26. Since any function in the attractor is a result of evolution by the semigroup, it locally satisfies the heat equation and consequently it is locally C^∞. In addition we can show that due to the differentiability of the velocity of the interface, functions in the attractor are C^3 up to the interface.

9. Evolution of the volume elements

In this section we present the principal ingredient of the proof of finiteness of the Hausdorff dimension of the attractor. The proof is based on a study of evolution of the infinitesimal volume along the orbits in the attractor (i.e., the evolution under the linearized flow). We demonstrate that for sufficiently large m that is defined solely by the physical parameters of the problem, the m-dimensional volume decays exponentially. This property combined with the compactness suggests that the Hausdorff dimension of the attractor is no larger than m. In the arguments regarding the Hausdorff dimension we follow quite closely the ideas outlined in [19, Chap. V]. The relation between the linearized flow and the fully nonlinear evolution is considered in Sec. 10.

9.1. *Linearized problem*

We treat the problem in the interface-attached coordinates. Let $U(.,t)$ be an orbit in the attractor, consider the linearization of (3.8)–(3.10) about U:

$$\begin{aligned} z_t &= z_{xx} + g(U(0,t))z_x + \beta z(0,t)U_x := F'(U)z, \\ [z_x(0,t)] &= \beta z, \quad z(x,0) = z_0(x) = U_0(x) - W_0(x), \qquad (9.53) \\ \text{where } \beta &= g'(U(0,t)) \qquad (9.54) \end{aligned}$$

Recall that $\beta < 0$, $|\beta| \le L$ where L is the Lipschitz constant for g, and that $v(t) = g(U(0,t)) < 0$ is the interface velocity.

First we note that the linearized problem is well-posed in the following sense:

THEOREM 27. *For any* $z_0 \in H_\omega$ *there exists a unique solution* z *of (9.53),* $z \in L^2(0,T;\Xi(t)) \cap C([0,T];H_\omega)$ *where* $\Xi(t) = \{f \in H^1_\omega,\ \beta f(0) + [f_x(0)] = 0\}$, *for any* $T > 0$.

Proof. This linear problem is somewhat nonstandard as it contains a nonlocal term $w(0,t)$. Nonetheless it can be handled as follows (cf. [17]). Consider first the problem (9.53) with a source, and zero initial conditions

$$\widetilde{w}_t - [\widetilde{w}_{xx} + \widetilde{w}_x g(U(0,t))] = f(x,t), \quad \beta\widetilde{w}(0,t) + [\widetilde{w}_x(0,t)] = 0, \quad \widetilde{w}(x,0) = 0,$$

We view the equation above as $\mathcal{L}\widetilde{w} = f(x,t)$; let $\mathcal{L}^{-1}$ be its solution operator: $\widetilde{w} = \mathcal{L}^{-1}f(x,t)$.

Then a solution of (9.53) is a superposition $z = \widetilde{w} + W$ of an appropriate $\widetilde{w}$ and of $W(x,t)$, which solves the *homogeneous* problem ($f(x,t) = 0$) with the initial condition $z_0(x)$. By inverting (9.53) we obtain an equation for $\widetilde{w}$:

$$\widetilde{w} + W = \mathcal{L}^{-1}\left[\beta U_x(x,t)(\tilde{w}(0,t) + W(0,t)\right]$$

Restricted to the boundary, it produces an integral equation for $\widetilde{w}(0,t)$:

$$\mathcal{L}^{-1}[-\widetilde{w}(0,t)U_x\beta + W(x,t)]_{x=0} = \tilde{w}(0,t) + W(0,t). \tag{9.55}$$

It is not difficult to show that the above equation is uniquely solvable as an integral equation with a sufficiently regular kernel. Thus, $\widetilde{w}$ is found and, consequently, the problem (9.53) can be solved locally in time.

The global existence can be obtained from the energy estimate:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|_\omega^2 &= -\beta z^2(0,t) - \|z_x\|_\omega^2 + (\alpha - v)\int e^{-ax} z z_x + \beta z(0,t)\int e^{-ax} z U_x \\
&\underset{\beta\le L}{\le} -\|z_x\|_\omega^2 + |\alpha - V_0|\left(\|z\|_\omega^2/\varepsilon_1 + \varepsilon_1\|z_x\|_\omega^2\right) \\
&\quad + L\left[\|z\|_\omega^2/\varepsilon^2 + \varepsilon^2\|z_x\|_\omega^2 - \alpha\|z\|_\omega^2\right]^{1/2}\|z\|_\omega\|U_x\|_\omega \\
&\le -\|z_x\|_\omega^2 + |\alpha - V_0|\left(\frac{1}{\varepsilon_1}\|z\|_\omega^2 + \varepsilon_1\|z_x\|_\omega^2\right) + L\left[\frac{1}{\varepsilon}\|z\|_\omega + \varepsilon\|z_x\|_\omega\right]\|z\|_\omega C_1
\end{aligned}$$

where C_1 is a bound on the H^1_ω-norm on the elements of the attractor. By choosing ε and ε_1 sufficiently small we can make the coefficient of $\|z_x\|_\omega^2$ equal to $-1/2$ and therefore obtain the usual energy inequality, $\frac{d}{dt}\|z\|_\omega^2 + \|z_x\|_\omega^2 \le C\|z\|_\omega^2$,which yields global existence in $L^2(0,T;\Xi(t)) \cap C([0,T];H_\omega)$. □

9.2. *Trace estimate*

Let $\xi_j,\ j = 1, ..., m$ be elements of H_ω and let z_j be the solution of the linearized problem with initial data ξ_j. Then the infinitesimal volume element spanned by $\{\xi_1, \dots, \xi_m\}$ evolves accordingly to the well-known formula

$$|z_1(t) \wedge \dots \wedge z_m(t)| = |\xi_1 \wedge \dots \wedge \xi_m| \exp \int_0^t Tr\,[F'(U(\tau)) \circ Q_m(\tau)]d\tau,$$

where $Q_m(\tau) = Q_m(\tau, U; \xi_1, \dots, \xi_m)$ is the projector onto the space $\Xi(\tau)$ spanned by $\{z_1(\tau), \dots, z_m(\tau)\}$. If ϕ be an element of $\Xi(\tau)$, then for the diagonal entry of the matrix of the m-dimensional projection we have

$$\begin{aligned}
\langle F'\phi, \phi\rangle_\omega &= \int \omega\phi_{xx}\phi dx + v(\tau)\int \omega\phi_x\phi dx + [\phi'(0)]\int \omega U_x\phi\, dx \\
&= -\|\phi_x\|_\omega^2 + (\alpha + v)\int e^{-\alpha x}\phi_x\phi dx + [\phi'(0)]\int \omega U_x\phi\, dx
\end{aligned} \tag{9.56}$$

It is easily seen via integration by parts that

$$2\int \omega\phi_x\phi dx = (\phi^2 e^{-\alpha x})|_0^\infty + (\phi^2 e^{-\alpha x})|_{-\infty}^0 + \int \omega\phi^2 dx = \alpha \tag{9.57}$$

Thus (9.56) becomes

$$\begin{aligned}\langle F'\phi,\phi\rangle_\omega &= -[\phi'(0)]\phi(0) - \|\phi_x\|_\omega^2 + [\phi'(0)]\int \omega U_x\phi dx + \alpha^2/2 + \frac{v\alpha}{2}\\ &\underset{|\beta|\le L}{\le} L\phi^2(0) - \|\phi_x\|_\omega^2 + \alpha^2/2 + \frac{v\alpha}{2} + \beta\phi(0)\int \omega U_x\phi dx \qquad (9.58)\end{aligned}$$

We select a special orthonormal basis in the subspace Ξ. We choose $m-1$ basis elements satisfying $\phi(0)=0$ (since $\phi(0)=0$ defines an $(m-1)$-dimensional subspace) and one with $\phi(0)\neq 0$. Next we estimate the principal negative term $-\|\phi_x\|_\omega^2$. Since for any $c>0$

$$2\int e^{-\alpha x}\phi_x\phi dx \le c\int e^{-\alpha x}\phi_x^2 dx + \frac{1}{c}\int e^{-\alpha x}\phi^2 dx$$

then

$$\|\phi_x\|_\omega^2 \ge \frac{2}{c}\int \omega\phi_x\phi dx - 1/c^2 \underset{(9.57)}{=} \frac{\alpha}{c} - 1/c^2 \underset{\text{Set } c=2/\alpha}{=} \alpha^2/4$$

If $\phi(0)=0$ then

$$\langle F'\phi,\phi\rangle_\omega = -\|\phi_x\|_\omega^2 + \alpha^2/2 + v\alpha/2 \le \alpha^2/4 - v_0\alpha/2 \qquad (9.59)$$

we remind that the velocity is negative $-V_0\le v\le -v_0<0$. Therefore *for* $\phi(0)=0$ *the matrix entry is negative* if $\alpha<2v_0$ (note that if $\alpha=v_0$ then $\langle F'\phi,\phi\rangle_\omega$ attains its minimum $-v_0^2/4$).

If $\phi(0)\neq 0$ we need to estimate the extra terms in (9.58) that contain $\phi(0)$ as a factor. First we note that

$$\begin{aligned}\left|\beta\phi(0)\int \omega U_x\phi dx\right| &\le \frac{1}{2b}\int \omega U_x^2 dx + \frac{b}{2}(\beta\phi(0))^2\int \omega\phi^2 dx\\ &\underset{|\beta|\le L}{\le} \frac{1}{2b}\|U_x\|_\omega^2 + \frac{b}{2}(L\phi(0))^2 \qquad (9.60)\end{aligned}$$

where $b>0$ will be chosen later. This yields:

$$\begin{aligned}\mu := \langle F'\phi,\phi\rangle_\omega &\underset{(9.60)}{\le} L\phi^2(0) - \|\phi_x\|_\omega^2 + \frac{\alpha^2}{2} + \frac{v\alpha}{2} + \frac{1}{2b}\|U_x\|_\omega^2 + \frac{b}{2}L^2\phi^2(0)\\ &\underset{(5.21)}{\le} \left(L + L^2\frac{b}{2}\right)\left(c\|\phi_x\|_\omega^2 - \alpha + \frac{1}{c}\right)\\ &\qquad - \|\phi_x\|_\omega^2 + \frac{1}{2b}\|U_x\|_\omega^2 + \frac{\alpha(\alpha-v_0)}{2}\\ &\underset{\alpha<v_0}{\le} \left[\left(L + L^2\frac{b}{2}\right)c - 1\right]\|\phi_x\|_\omega^2 + \left(L + L^2\frac{b}{2}\right)\frac{1}{c} + \frac{1}{2b}\|U_x\|_\omega^2\end{aligned}$$

To eliminate the term with $\|\phi_x\|^2_\omega$, which we cannot control, we select $b = 2/L$, $c = 1/(2L)$ and obtain

$$\mu \leq 4L^2 + (L/4)\,\|U_x\|^2_\omega \underset{(8.52)}{\leq} 4L^2 + (L/4)\,B(\alpha)$$

Thus the *diagonal entry for the basis element with* $\phi(0) \neq 0$ *is bounded above.*

If for definiteness α is taken as $\alpha = \lambda v_0$, $0 < \lambda < 1$, then

$$\mu \leq C(\lambda) V_0^2 L/(4v_0^3) + 4L^2 \tag{9.61}$$

where $C(\lambda) = O(1)$. In the leading order, assuming $v_0 \ll 1$,

$$\mu \leq CLV_0^2/v_0^3, \tag{9.62}$$

where C is an absolute constant. Employing the above estimates for the trace entries (9.59), (9.61) we complete the estimate for the trace

$$Tr\,[F'(U(\tau), V(\tau)) \circ Q_m(\tau)] = \sum_{i=1}^{m} \langle F'\phi_i, \phi_i\rangle_\omega \leq \mu - (m-1)v_0^2/4$$

In the leading order,

$$m \underset{(9.62)}{>} CLV_0^2/v_0^5 \tag{9.63}$$

is sufficient for the trace to become negative.

10. Differentiability of the semigroup

To utilize the trace estimate developed in the previous section we need to demonstrate that the nonlinear evolution of the volume is well approximated by the evolution generated by the linearization (9.53). This will be ensured by the theorem on differentiability of the evolution operator with respect to the initial conditions (cf. [19, Sec. V.3.3]):

THEOREM 28. *Let* U *and* W *be two orbits* $U = T(t)U_0$, $W = T(t)W_0$, $U_0, W_0 \in H_{\omega,1}$ *and* $z(t)$ *be a solution of the linearized problem (9.53) with the initial condition* $z(0) = U_0 - W_0$. *Then for any* t_0,

$$\|U(t) - W(t) - z(t)\|_\omega \leq const\;\; \|U_0 - W_0\|^{3/2}_\omega$$

where $0 \leq t \leq t_0$ *and the constant depends only on* t_0.

Note that the mapping $z(0) \to z(t)$ is the Frechét differential of $T(t)$ at U_0.

Proof. First we note that by Proposition 5 the interface temperature is uniformly bounded, and for solutions in the attractor (4.20) yields

$$U(0,t) \le V_0/v_0. \tag{10.64}$$

Then since $g(U)$ is strictly monotone, $g(U) \le V_M := g(V_0/v_0) < V_0$. The function $(g^{-1})'(v)$ is bounded on the interval $[v_0, V_M]$, consequently $g'(U(0,t)) \ge d_1$. Also we assume $g \in C^2$ and therefore $|g''|$ is bounded on $[0, V_0/v_0]$ by some d_2.

It is easy to see that the difference $w = U - W$ solves the following problem

$$\begin{aligned} w_t &= w_{xx} + g(U(0,t))w_x - [w_x(0,t)]w_x + [w_x(0,t)]U_x, \\ [w_x(0,t)] &= -\Delta v = g(U(0,t)) - g(W(0,t)) = g'(\theta)w(0,t) \\ w(x,0) &= U_0(x) - W_0(x). \end{aligned} \tag{10.65}$$

which is similar to (7.45). Let $z(x,t)$ be a solution of the linearization in (9.53). Then the difference $y = w - z$ is a solution of the following problem

$$\begin{aligned} &y_t = y_{xx} + g(U(0,t))y_x + [y_x(0,t)]U_x - [w_x(0,t)]w_x, \qquad (10.66)\\ &[y_x]_{x=0} = g'(\theta)w|_{x=0} - g'(U(0,t))z|_{x=0} = [g'(\theta) - g'(U)]w|_{x=0} + g'(U)y|_{x=0} \\ &y(x,0) = 0 \end{aligned}$$

where θ is between $U(0,t)$ and $W(0,t)$. We denote $G := \left\{g'(\theta) - g'(U)\right\} w|_{x=0}$, then the boundary condition in (10.66) takes the form

$$y(0,t) = \{[y_x]|_{x=0} - G\} / g'(U(0,t)) \tag{10.67}$$

We multiply the equation in (10.66) by ωy and integrate by parts to obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y\|_\omega^2 = -[y_x]y|_0 - \|y_x\|_\omega^2 + \alpha \int \omega y_x y dx \\ &\quad +[U_x(0,t)] \int \omega y_x y dx + [y_x(0,t)] \int \omega U_x y dx - [w_x(0,t)] \int \omega w_x y dx \end{aligned} \tag{10.68}$$

For the first term above we have

$$\begin{aligned} &|[y_x(0,t)]\, y(0,t)| \underset{(10.67)}{=} \left|G y(0,t) + g' y^2(0,t)\right| \\ &\quad \le \{G^2 + y^2(0,t)\}/2 + \left|g'\right| y^2(0,t) \\ &\quad \underset{\text{Lemma 7}}{\le} (\left|g'\right| + 1/2) + G^2/2 \end{aligned}$$

Next, $\alpha \int \omega y_x y dx \le \alpha/2$. Similarly,

$$\left|[U_x(0,t)] \int \omega y_x y dx\right| \underset{|[U_x]| \le V_0}{\le} (V_0/2)\,(\|y\|_\omega^2/\varepsilon + \varepsilon\,\|y_x\|_\omega^2)$$

For the next to the last term in (10.68) we obtain

$$
\begin{aligned}
&\left|[y_x(0,t)]\int \omega U_x y dx\right| \\
&\underset{\text{Cauchy-Schwartz}}{\leq} (|g'||y(0,t)| + |G|)\int \omega |U_x y| dx \leq (|g'||y(0,t)| + |G|)\ \|y\|_\omega \|U_x\|_\omega \\
&\underset{\text{Lemma 7}}{\leq} \{|g'|\left(\frac{1}{\varepsilon}\|y\|_\omega^2 + \varepsilon\|y_x\|_\omega^2\right)|^{1/2} + |G|\}\|y\|_\omega \|U_x\|_\omega \\
&\leq \{|g'|(\sqrt{1/\varepsilon}\|y\|_\omega + \sqrt{\varepsilon}|\|y_x\|_\omega) + |G|\}\|y\|_\omega \|U_x\|_\omega \\
&\leq |g'|\sqrt{1/\varepsilon}\|y\|_\omega^2 \|U_x\|_\omega + \frac{|g'|^2}{2}\sqrt{\varepsilon}\|y_x\|_\omega^2 + \frac{1}{2}\|y\|_\omega^2 \|U_x\|_\omega^2 \\
&\quad + \frac{1}{2}\|y\|_\omega^2 + \frac{1}{2}G^2\|U_x\|_\omega^2 \\
&:= |g'|^2\sqrt{\varepsilon}\|y_x\|_\omega^2/2 + A\|y\|_\omega^2 + B^2G^2/2
\end{aligned}
$$

where B is an upper bound on $\|U_x\|_\omega$ [cf. (8.52)], and $A = A(\varepsilon) > 0$.

Finally for the last term in (10.68)

$$
\begin{aligned}
&\left|\int \omega w_x[w_x(0,t)]y dx\right| \leq |[w_x(0,t)]|\ \|w_x\|_\omega \|y\|_\omega \\
&\leq \frac{c}{2}\|w_x\|_\omega^2 + \frac{1}{2c}|[w_x(0,t)]|^2\|y\|_\omega^2 \\
&= \frac{c}{2}\|w_x\|_\omega^2 + \frac{1}{2c}\|y\|_\omega^2|\Delta v|^2 \underset{(7.44)}{\leq} \frac{c}{2}\|w_x\|_\omega^2 + \frac{1}{c}\|y\|_\omega^2 C_1\|w_0\|_\omega^2 t^{-1/2}
\end{aligned}
$$

Upon selecting $c = \|w_0\|_\omega^2$ we estimate this term as

$$
\left|\int \omega w_x[w_x(0,t)]y dx\right| \leq (1/2)\|w_x\|_\omega^2\|w_0\|_\omega^2 + C_1 t^{-1/2}\|y\|_\omega^2
$$

Collecting the estimates for all the terms in (10.68) we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y\|_\omega^2 &\leq (g' + 1/2)\left(\frac{1}{\varepsilon}\|y\|_\omega^2 + \varepsilon\|y_x\|_\omega^2\right) + G^2/2 - \|y_x\|_\omega^2 \\
&+ \frac{\alpha}{2}\left(\frac{1}{\varepsilon}\|y\|_\omega^2 + \varepsilon\|y_x\|_\omega^2\right) + V_0\left(\frac{1}{\varepsilon}\|y\|_\omega^2 + \varepsilon\|y_x\|_\omega^2\right) \\
&+ \frac{|g'|^2}{2}\sqrt{\varepsilon}\|y_x\|_\omega^2 + A\|y\|_\omega^2 + \frac{B^2}{2}G^2 + \frac{1}{2}\|w_x\|_\omega^2\|w_0\|_\omega^2 + C_1 t^{-1/2}\|y\|_\omega^2
\end{aligned}
\tag{10.69}
$$

Now we can select ε sufficiently small so that the combined coefficient by $\|y_x\|_\omega^2$ is nonpositive. We collect the like terms in (10.69), drop the nonpositive term with $\|y_x\|_\omega^2$ and rearrange to obtain

$$
\frac{1}{2}\frac{d}{dt}\|y\|_\omega^2 - (C + C_1 t^{-1/2})\|y\|_\omega^2 \leq \frac{B^2}{2}G^2 + \frac{1}{2}\|w_x\|_\omega^2\|w_0\|_\omega^2.
$$

Recall that $|g''| \leq d_2$, hence

$$|G|^2 = |[g'(\theta) - g'(U(0,t))]w|^2 \leq d_2^2|\theta - U(0,t)|^2|w(0,t)|^2 \leq d_2^2|w(0,t)|^4$$

Note that $|w(0,t)|$ is uniformly bounded. As follows from (8.51), on the attractor $\sup|w(0,t)| \leq 2V_0/v_0 = C_0$. Therefore $|w(0,t)|^4 \leq C_0|w(0,t)|^3$. We continue the estimate as follows

$$|G|^2 \leq C_1|w(0,t)|^3 \underset{(10.65)}{\leq} C_1 d_1^{-3}|\Delta v|^3 \underset{(7.44)}{\leq} C \|w_0\|_\omega^3 t^{-3/4}$$

As a result we arrive at the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|y\|_\omega^2 - (C + C_1 t^{-1/2})\|y\|_\omega^2 \leq E\|w_0\|_\omega^3 t^{-3/4} + \frac{1}{2}\|w_x\|_\omega^2\|w_0\|_\omega^2. \tag{10.70}$$

By the Gronwall lemma,

$$\|y\|_\omega^2 \leq \exp(Ct + (2/3)\,C_1 t^{1/2})\left(E\|w_0\|_\omega^3\int_0^t t^{-3/4}d\tau + \frac{1}{2}\|w_0\|_\omega^2\int_0^t\|w_x\|_\omega^2\,d\tau\right)$$

$$\underset{(7.49)}{\leq} \exp(Ct + \frac{2}{3}C_1 t^{1/2})(4E\|w_0\|_\omega^3 t^{1/4} + \frac{1}{2}\|w_0\|_\omega^4\exp(Ct)) \underset{\|w_0\|_\omega<1}{\leq} c\|w_0\|_\omega^3$$

where c is a bound for the time dependent factor for $t \leq T$. □

Finally, the estimate for the dimension of the linear volume element and differentiability of the semigroup yield the principal result of the paper:

THEOREM 29. *The Hausdorff dimension of the attractor $\mathcal{A}$ is finite,*

$$\dim_H \mathcal{A} \leq CLV_0^2/v_0^5$$

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REFERENCES

[1] BRAUNER, C.-M., HULSHOF, J. and LUNARDI, A., *A general approach to stability in free boundary problems*, J. Differential Equations *164* (2000), 16–48.

[2] BRAUNER, C.-M. and LUNARDI, A., *Instabilities in a two-dimensional combustion model*, Arch. Ration. Mech. Anal. *154*, (2000) 157–182.

[3] CHEN, X. and REITICH, F., *Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling*, J. Math. Analysis Appl. *164* (1992) 350–362.

[4] DIN, S., TAO, Y. and YIN, H.-M., *A chemical diffusion process with reaction at free boundary*, Canad. Appl. Math. Quarterly *5* (1997) 49–74.

[5] Frankel, M. L. and Roytburd, V., *A free boundary problem modeling thermal instabilities: II: stability and bifurcation*, J. of Dyn. Sys. and Diff. Eqs *6* (1994), 447–486.

[6] Frankel, M., Qu, M. and Roytburd, V., *On a free interface problem modeling solid combustion and rapid solidification in infinite medium*, *World Scientific Series in Applicable Analysis, Vol. 4: Dynamical Systems and Applications*, R. P. Agarwal ed., World Scientific (1995), 263–278.

[7] Frankel, M., Roytburd, V. and Sivashinsky, G., *Complex dynamics generated by a sharp interface model of self-propagating high-temperature synthesis*, Combust. Theory Modelling, *2* (1998), 479–496.

[8] Frankel, M. and Roytburd, V., *On attractors for a sharp interface model of exothermic phase transitions*, Adv. Math Sci. Appl. *14* (2004) 25–40.

[9] Frankel, M. and Roytburd, V., *Finite-dimensional attractors for a nonequilibrium Stefan problem with heat losses*, DCDS *13* (2005), 35–62.

[10] Frankel, M. and Roytburd, V., *Dynamics of thermally-insulated nonequilibrium Stefan problem*, Preprint (2006), http://www.rpi.edu/˜roytbv/papers/FrankelRoytburd.pdf

[11] Gol'berg, S. M. and Tribelskii, M. I., *On laser induced evaporation of nonlinear media*, Zh. Tekh. Fiz. (Sov. Phys.-J. Tech. Phys.) *55* (1985), 848–857.

[12] Lorenzi, L., *A free boundary problem stemmed from combustion theory*. Part I, J. Math. Anal. Appl. *274* (2002), 505–535.

[13] Luckhaus, S., *Solutions for the two-phase Stefan problem with the Gibbs–Thomson law*, Euro. J. of Appl. Math. *1* (1990), 101–111.

[14] Matkowsky, B.J. and Sivashinsky, G. I., *Propagation of a pulsating reaction front in solid fuel combustion*, SIAM J. Appl. Math. *35* (1978), 230–255.

[15] Munir, Z. A. and Anselmi-Tamburini, U., *Self-propagating exothermic reactions*, Mater. Sci. Rep. *3* (1989), 277–365.

[16] Radkevich, E. V., *The Gibbs–Thomson correction and the classical existence for the modified Stefan problem*, Soviet Math. Dokl. *41* (1991), 274–278.

[17] Roytburd, V., *A Hopf bifurcation for a reaction-diffusion equation with concentrated chemical kinetics*, J. Diff. Eqs., *56* (1985), 40–62.

[18] Shkadinsky, K. G., Khaikin, B. I. and Merzhanov, A. G., *Propagation of a Pulsating Exothermic Reaction Front in the Condensed Phase*, Combust. Expl. Shock Waves *7* (1971), 15–22.

[19] Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer Verlag: New York (1988).

[20] Van Saarloos, W. and Weeks, J., *Surface undulations in explosive crystallization: a nonlinear analysis of a thermal instability*, Physica D *12* (1984), 279–294.

[21] Varma, A. Rogachev, A. S., Mukasyan, A. S. and Hwang, S., *Combustion synthesis of advanced materials*, Adv. Chem. Eng. *24* (1998), 79–226.

[22] Varma, A., *Form from fire*, Sci. American Aug. (2000), 58–61.

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