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Dissipative Dynamics for a Class of Nonlinear Pseudo-Differential Equations

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Abstract. We study a class of nonlinear evolutionary equations generated by an elliptic pseudo-differential operator, and with nonlinearity of the form $G(u_x)$ where $c\eta^2 \leq G(\eta) \leq C\eta^2$ for large $|\eta|$.

For the evolution in spaces of periodic functions with zero mean we demonstrate existence of a universal absorbing set and compact attractor. Furthermore, we show that the attractor is of a finite Hausdorf dimension. The dissipation mechanism for the class of equations studied in the paper is akin to the nonlinear saturation in the Kuramoto-Sivashinsky equation. A similar generalization of the Kuramoto-Sivashinsky equation was studied by Nicolaenko *et al.* under the assumption of a purely quadratic nonlinearity and reflection invariance of both: the equation and solutions.

1. Introduction

Consider a nonlinear evolution equations of the form

$$u_t + G(u_x) = \mathcal{P}(D)u \tag{1.1}$$

where the linear part $\mathcal{P}(D)$ is an elliptic pseudo-differential operator of the order $2m$. The function $G(\eta)$ is assumed to be sufficiently smooth and satisfying the condition: $c_{\min}\eta^2 \leq G(\eta) \leq C_{\max}\eta^2$, for $|\eta|$ large.

In some sense (1.1) may be regarded as a generalization of the Kuramoto-Sivashinsky equation [12, 11, 15]

$$u_t + \frac{1}{2}u_x^2 = -u_{xx} - u_{xxxx} \tag{1.2}$$

A similar generalization was studied by Nicolaenko *et al.* [13] under the assumption of invariance of the equation (and solutions) with respect to the reflection $x \to -x$, and for a purely quadratic nonlinearity $G(\eta) = C\eta^2$.

In the current paper we demonstrate that the reflection invariance condition of [13] can be relaxed, and the nonlinearity can be of a more general form without destroying the dissipativity of the system. For the broader class of equations defined in (1.1) (more

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precisely for its differentiated with respect to x version (2.1)) we demonstrate existence of a global absorbing set, which, as it is often the case for dissipative systems, implies existence, compactness, and finite Hausdorff dimensionality of an attractor in the spaces of periodic functions with zero mean. In our study we employ a technique developed in [2], which combines the ideas introduced by Collet et al. [5] and Goodman [8].

Although we allow the nonlinearity in (1.1) to be of a more general form than the conventional purely quadratic function, it should be remarked that, in our view, even the requirement of quadratic growth of G at infinity should be attributed to our technical approach rather than to the intrinsic nature of the problem.

Equations of type (1.1) may generate a variety of dynamical patterns as a result of interplay between the unstable modes in the spectrum of linear operator $\mathcal{P}$ and dispersion of "energy" over the entire spectrum due to the nonlinearity. As we demonstrate below, a sufficient amount of energy is then dissipated by the short-wave spectrum for the solutions to uniformly gravitate to the absorbing set in appropriately chosen norms.

The family of equations (1.1) includes, for example, the Kawahara equation [18] (a.k.a. the generalized Kuramoto-Sivashinsky equation)

$$u_t + \frac{1}{2}u_x^2 = -u_{xx} + \gamma u_{xxx} - u_{xxxx},$$

which was introduced as a model equation for surface waves on a viscous film. The presence of the linear dispersion obviously leads to different dynamical patterns generated by this equation.

Another instance of the equation (1.1) is the nonlocal evolution equation modelling cellular flames

$$u_t + \frac{1}{2}u_x^2 = u_{xx} + \alpha(I - A)u,$$

where

$$A := (I - \partial_x^2)^{-1},$$

In a recent papers [1] this equation was shown to generate a cellular-chaotic evolution.

Yet another representative of the family of equations (1.1) is the nonlocal Kuramoto-Sivashinsky equation (see [15, 10]) describing evolution of a weakly perturbed plane flame front in the limit of a relatively small thermal expansion of the combustible gas.

At the core of the paper is the proof of dissipativity, i.e. of the uniform boundedness of solutions originating in a certain ball of initial data (Theorem 3.4). The proof employs a variant of Gårding's inequality for the linear part and a Poincaré type inequality to control the behavior of the nonlinear part in the energy estimates. The diameter of the absorbing set in the H^0-norm is polynomial in the period L.

The bound in H^0 can be extended to the bounds in all H^s with $0 < s \leq m$; however these bounds are exponential in the period L, except for the case of the purely quadratic nonlinearity where it is still polynomial. Stability in all H^s with $0 < s \leq m$ automatically guarantees compactness of the dynamical system under consideration in any H^s with $0 \leq s < m$ thanks to the precompactness of the imbedding in Sobolev spaces.

Next we investigate the evolution of the infinitesimal volume on the attractor and, based on the differentiability of the corresponding semigroup, conclude that the Hausdorff dimension of the attractor is finite. Depending on the norms used in the dimension estimates the results differ in regard to their behavior as the period increases. If the dimension is estimated in the sense of H^0 then a polynomial in L growth can be obtained for a more general (than the purely quadratic) form of G. The value of the dimension in the sense of H^k, $1 \leq k \leq m - 1$ is exponential in L, with the exception once again of the purely quadratic nonlinearity where it is still polynomial. Technically the dimension calculation follows a rather well-established path of the trace estimation (see e.g. the book of Temam [17]); however, opting for the transparency of presentation, we did not attempt to obtain necessarily the lowest possible estimate (cf. [7, 3, 14] for the Kuramoto-Sivashinsky equation, see also [4]).

2. Statement of the problem and preliminaries

To eliminate the drift due to the nonzero mean, it is more convenient (and rather conventional) to study the initial-value problem for the equation (1.1) differentiated with respect to x

$$\begin{aligned} u_t + [G(u)]_x &= \mathcal{P}(D)u \\ u(x,0) &= u_0(x), \quad u_0 \in \dot{H}^0_{per} \end{aligned} \tag{2.1}$$

in the Sobolev spaces of L-periodic functions with *zero mean* denoted by $\dot{H}^s_{per}$ (we abuse notation and keep the letter u for the dependent variable).

Recall that in the periodic case, the usual Sobolev norm is defined through the Fourier series as

$$||u||_s^2 = \sum_{k=-\infty}^{\infty} \left(1 + \xi_k^2\right)^s |\tilde{u}_k|^2 = \int_{-L/2}^{L/2} \left| \left(1 - \frac{d^2}{dx^2}\right)^{s/2} u(x) \right|^2 dx$$

here $\xi_k = 2\pi k/L$ is the spectral parameter. Note that for $u \in \dot{H}^s_{per}$, $\tilde{u}_0 = 0$ and therefore the norm is equivalent to the following leading order norm:

$$|u|_s^2 = \sum_{k=-\infty}^{\infty} |\xi_k|^{2s} |\tilde{u}_k|^2$$

For simplicity of presentation we assume that $\mathcal{P}$ is an elliptic pseudo-differential operator with the symbol $p(\xi)$,

$$\begin{gathered} p(\xi) = -C\,|\xi|^{2m} + p_1(\xi), \quad m \geq 1 \\ |p_1(\xi)| \leq c\,|\xi|^{\tilde{m}} \text{ for } |\xi| \text{ large}, \quad \tilde{m} < 2m \end{gathered} \tag{2.2}$$

$\mathcal{P}$ is assumed to preserve the subspace of the zero-mean functions.

REMARK 2.1. It is not difficult to modify all the proofs below for *any classical elliptic pseudo-differential operator with periodic coefficients* such that it preserves the space of zero-mean functions. It is also not difficult to extend somewhat the class of acceptable operators to the operators of the form

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$$

where $\mathcal{P}_1$ is an abstract linear operator (not necessarily pseudo-differential one) in $\dot{H}^0_{per}$ that satisfies the inequality

$$|\langle \mathcal{P}_1 u, u\rangle| \leq \rho\,\|u\|^2_{H^{\tilde{m}}}$$

here $\langle .\,,.\rangle$ is the H^0 inner product (cf. [13], where however because of the specifics of the proof, the operator is required to be an even function of D alone). Again $\mathcal{P}_1$ is assumed to preserve the space of zero-mean functions.

We note that ellipticity of the linear operator allows us to use the general result due to Gårding (see, for example, Taylor [16, Chap. 7, Theorem 6.1]) that for our purposes can be stated as follows.

THEOREM 2.2. *Let operator $\mathcal{P}$ be of the type defined in (2.2). Then, there exist constants c_0, α such that for $u \in H^m_{per}(L)$,*

$$\operatorname{Re}\langle \mathcal{P}u, u\rangle \leq -c_0\,\|u\|^2_{H^m} + \alpha\,\|u\|^2. \tag{2.3}$$

Proof. For our case the proof is straightforward:

$$\begin{aligned} \operatorname{Re}\langle p(D)u, u\rangle &= \operatorname{Re}\left[\langle p_0(i\xi)\tilde{u}, \tilde{u}\rangle + \langle p_1(i\xi)\tilde{u}, \tilde{u}\rangle\right] \\ &\leq -C\,\left\| |\xi|^m\,\tilde{u}\right\|^2 + \operatorname{Re}\langle p_1(i\xi)\tilde{u}, \tilde{u}\rangle \\ &= -C\left\|\left(1+|\xi|^2\right)^{m/2}\tilde{u}\right\|^2 \\ &\quad + \operatorname{Re}\left\langle\left[C\left(|\xi|^m - \left(1+|\xi|^2\right)^{m/2}\right)^2 + p_1(i\xi)\right]\tilde{u}, \tilde{u}\right\rangle \\ &:= -C\left\|\left(1+|\xi|^2\right)^{m/2}\tilde{u}\right\|^2 + \operatorname{Re}\langle p_2(i\xi)\tilde{u}, \tilde{u}\rangle \\ &\leq -C\,\|u\|^2_{H^m} + \left\|(c\left(1+|\xi|^2\right)^{\max(\tilde{m}, m-1)/2} + c_1)\tilde{u}\right\|^2 \\ &\leq -C\,(1-\varepsilon)\,\|u\|^2_{H^m} + \alpha\,\|u\|^2 := -c_0\,\|u\|^2_{H^m} + \alpha\,\|u\|^2. \end{aligned}$$

We note that α depends on the choice of ε; for the sequel ε is fixed, say $\varepsilon = 1/2$. In the calculation above the tilde stands for the discrete Fourier transform, while ξ is also discrete $\xi_n = 2n\pi/L$. □

Below for simplicity of presentation (to avoid dealing with real parts) we shall assume that $\mathcal{P}$ is a *real operator*.

The nonlinearity in (2.1) is assumed to satisfy

$$\begin{aligned} c_{\min}\eta^2 \le G(\eta) &\le C_{\max}\eta^2, \text{ for } |\eta| \text{ large} \\ \left|G'(\eta)\right| &\le M_G\,|\eta|\,, \text{ for } |\eta| \text{ large} \end{aligned} \tag{2.4}$$

for some *positive* constants $c_{\min}$, $C_{\max}$, and M_G.

The assumption in (2.4) is valid for sufficiently *large* $|\eta|$; it is easy to show that *for all* η, G can be represented as a sum,

$$\begin{gathered} G(\eta) = G_0(\eta) + G_1(\eta), \quad c_{\min}\eta^2 \le G_0(\eta) \le C_{\max}\eta^2, \\ \operatorname{supp} G_1 < \infty, \quad |G_1(\eta)| \le \gamma_1. \end{gathered} \tag{2.5}$$

Indeed, let M be such that for $|\eta| > M$, $c_{\min}\eta^2 \le G(\eta) \le C_{\max}^2\eta$. Then $G(\eta) = G_0(\eta) + G_1(\eta)$ where $G_0 = G$ for $|\eta| > M$ and is a smooth extension of G into the domain $|\eta| \le M$ with the bounds $c_{\min}\eta^2 \le G_0(\eta) \le C_{\max}\eta^2$; while $G_1(\eta) = G(\eta) - G_0(\eta)$ for all η. Obviously, $\operatorname{supp} G_1 \subset [-M, M]$ and $|G_1(\eta)| \le \gamma_1$ for some γ_1.

The decomposition of G in (2.5) is done so that we are able to deal with a function that is $\mathcal{O}(\eta^2)$ throughout the entire domain in the proof of the uniform estimate below. It is merely a matter of technical convenience and can be dealt with via a different argument. We note, however that in our proof we use both $c_{\min}$ and $C_{\max}$.

Local existence for the parabolic initial-value problem in (2.1) is quite routine (see e.g. [9, 17]). Clearly the principal part of $\mathcal{P}(D)$ defines a sectorial operator. Since the nonlinearity is locally Lipschitz from $H^1 \to H^0$, local well-posedness follows by analytic semigroup methods. This and the a priori energy estimates allow one to incur the global existence in $L^\infty([0,T); \dot{H}^0_{per}) \cap L^2(0,T; \dot{H}^1_{per})$ for any T:

THEOREM 2.3. *Let u be a solution of (2.1); then for any $T > 0$, $u \in L^\infty([0,T); \dot{H}^0_{per}) \cap L^2(0,T; \dot{H}^m_{per})$.*

Proof. First it is easy to obtain the usual energy estimate. We multiply the equation by u and integrate

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \langle \mathcal{L}u, u\rangle - \int u_x G'(u)u \underset{\text{Gårding}}{\le} -c_0\left\|\partial^m u\right\|^2 + \alpha\,\|u\|^2\,, \tag{2.6}$$

note that the nonlinear term integrates to 0. It immediately yields

$$\|u\|^2 \leq \|u_0\|^2 \exp(2\alpha t) \tag{2.7}$$

(it also secures global existence in $\dot{H}^0_{per}$). The inequality in (2.6) can be rewritten in the form

$$c_0 \left\|\partial^m u\right\|^2 \leq -\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 \tag{2.8}$$

By integrating the estimate (2.8) from 0 to T, we see that

$$\int_0^T \left\|\partial^m u\right\|^2 dt \leq \frac{\alpha}{c_0}\int_0^T \|u\|^2 dt + \frac{1}{2c_0}\left(\|u(T)\|^2 + \|u_0\|^2\right) \leq \frac{1}{c_0}\exp(2\alpha t)\|u_0\|^2$$

which leads to the desired conclusion. ☐

REMARK 2.4. A rather conventional bootstrap argument allows one to conclude that if $u_0 \in \dot{H}^s_{per}$ with $s \geq 1$ then $u \in L^\infty([0,T); \dot{H}^s_{per}) \cap L^2 0, T; \dot{H}^{m+s}_{per})$ (cf. [1], see also Remark 4.5). Moreover, the solution is infinitely smooth (and in fact analytic in time) for $t > 0$.

Note, however, that the estimates above allow for exponential growth of the norms with T. The purpose of the next section is to prove uniform boundedness of solutions $u \in L^\infty([0,\infty); \dot{H}^0_{per})$.

3. Absorbing set

In this section we establish the existence of an absorbing set for the initial-boundary value problem (2.1). In the spirit of [8, Proposition 1] we will employ the following Poincaré type result:

LEMMA 3.1. *Let $b \in C^\infty$ be the Sobolev's mollifier*

$$b(x) = \begin{cases} 0, |x| > \varepsilon \\ \dfrac{a_0}{\varepsilon} B \exp\left[-\dfrac{1}{1-(x/\varepsilon)^2}\right], |x| \leq \varepsilon \end{cases}$$

where

$$a_0 = \left(\int_{-1}^{1} \exp\left[\frac{-1}{1-x^2}\right] dx\right)^{-1}$$

(thus making the area under the bell-curve equal B). Then for any $u \in H^1$

$$\int b(x)u^2(x)dx \leq \frac{4}{B}\langle ub\rangle^2 + B\varepsilon' \int u'(x)^2 dx$$

where $\varepsilon' = 32\varepsilon\,(a_0/e)^2$, $\langle ub\rangle = \int b(y)u(y)dy$.

Proof. On the interval $x, y \in [-\varepsilon, \varepsilon]$

$$u(x) = u(y) + \int_y^x u'(z)dz$$

We multiply this equation by $b(y)$ and integrate with respect to y

$$\begin{aligned} Bu(x) &= \langle ub\rangle + \int_{-\varepsilon}^{\varepsilon} b(y)dy \int_y^x u'(z)dz \\ &:= \langle ub\rangle + \int_{-\varepsilon}^{\varepsilon} K(x,z)u'(z)dz \\ &:= \langle ub\rangle + \hat{K}u' \end{aligned}$$

Here $\hat{K}$ is an integral operator on $L^2[-\varepsilon, \varepsilon]$ with the kernel

$$K(x,z) = \begin{cases} \int_z^{\varepsilon} b(y)dy, & \text{if } z > x \\ \int_{-\varepsilon}^{z} b(y)dy, & \text{if } z < x \end{cases}$$

Next we calculate

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} u^2(x)dx &= \frac{1}{B^2}\int_{-\varepsilon}^{\varepsilon} [\langle ub\rangle^2 + 2\langle ub\rangle(Ku')(x) + (Ku')^2(x)]dx \\ &\leq \frac{2}{B^2}\int_{-\varepsilon}^{\varepsilon} [\langle ub\rangle^2 + (Ku')^2(x)]dx = \frac{2}{B^2}\left[2\varepsilon\langle ub\rangle^2 + \int_{-\varepsilon}^{\varepsilon}(Ku')^2(x)dx\right] \\ &\leq \frac{4\varepsilon}{B^2}\langle ub\rangle^2 + \frac{2}{B^2}||\hat{K}||^2 \int_{-\varepsilon}^{\varepsilon} (u')^2(x)]dx \end{aligned}$$

It is clear that $|K(x,z)| \leq 2a_0 e^{-1} B$. Thus, the Hilbert-Schmidt norm

$$||K||^2 = \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |K(x,z)|^2 dxdz \leq 16\left(a_0 e^{-1}\right)^2 B^2\varepsilon^2$$

Therefore we get

$$\int b(x)u^2(x)dx \leq \sup b \int_{-\varepsilon}^{\varepsilon} u^2(x)dx \leq \frac{B}{\varepsilon}\left[\frac{4\varepsilon}{B^2}\langle ub\rangle^2 + \frac{2}{B^2}||\hat{K}||^2 \int_{-\varepsilon}^{\varepsilon}(u')^2(x)]dx\right]$$
$$= \frac{4}{B}\langle ub\rangle^2 + 32\,(a_0/e)^2\, B\varepsilon \int u'(x)^2 dx$$

□

REMARK 3.2. This inequality can be trivially extended for $G(u) \leq C_{\max} u^2$

$$\int b(x)G(u(x))dx \leq C_{\max}\int b(x)u^2(x)dx \leq C_{\max}\left(\frac{4}{B}\langle ub\rangle^2 + B\varepsilon'\int u'(x)^2 dx\right) \quad (3.1)$$

REMARK 3.3. It follows easily from the scaling of $b(x)$ that for any real $k \geq 0$

$$\left\|D^k b\right\|^2 \leq CB^2/\varepsilon^{2k+1} \quad (3.2)$$

where for non-integer k the fractional derivative operator is defined through the Fourier series. This estimate will be needed for the estimate on the absorbing set below.

Now we are ready to state the main stability result of this section, which is the following a priori estimate:

THEOREM 3.4. *(Existence of an absorbing ball). For any solution u of (2.1)*

$$\limsup_{t\to\infty}||u(.,t)|| \leq \begin{cases} C_a L^{2m+3/2} \text{ if } L > L_0 := 4C_{\max}c_{\min}/(\alpha+1) \\ C_a L_0^{2m+3/2} \text{ if } L \leq L_0 \end{cases}.$$

where C_a is a universal constant; $||.||$ is the H^0-norm.

Proof. Let s be a fixed, sufficiently smooth periodic function with 0 mean (to be selected later). Introduce $y(t)$ as a solution of the initial-value problem for the ordinary differential equation

$$\dot{y}(t) = \int u(x,t)s'(x+y(t)))dx,\ \ y(0) = 0.$$

where $u(x,t)$ is a given solution of (2.1). Consider the curve of all the shifts $s(.+\eta)$ of s, $0 \le \eta \le L$, and introduce the squared distance to the curve,

$$\Phi(t) := \frac{1}{2} \int_{-L/2}^{L/2} [u(x,t) - s(x+y(t))]^2 dx$$

It is easy to see that for the shift y_0, for which $\Phi(t)$ is minimal $\int u(x)s'(x+y_0) = 0$.

We compute the derivative of Φ and substitute u_t from the equation (2.1) to obtain

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \int_{-L/2}^{L/2} (u-s)\left[-\dot{y}s'(x+y(t)) - G'(u)u_x + \mathcal{P}u\right] \\ &= \int u\mathcal{P}(u) + \int sG'(u)u_x - \dot{y}\int us' - \int s\mathcal{P}u \qquad (3.3) \end{aligned}$$

note that both $\int ss' = 0$ and $\int G'(u)u_x u = 0$. In the calculation above and everywhere in the sequel we keep the notation s for the shifted function $s(.+y(t))$; we also use s_0 for the unshifted $s(.+0)$.

We insert the definition of $\dot{y}$ into (3.3) and estimate $d\Phi/dt$:

$$\begin{aligned} \frac{d}{dt}\Phi(t) &\underset{\text{Gårding}}{\le} -c_0\,\|u\|^2_{H^m} + \alpha\,\|u\|^2 - \langle us'\rangle^2 + \int sG'(u)u_x - \int u\mathcal{P}^*s \\ &\le -c_0\,\|u\|^2_{H^m} + \alpha\,\|u\|^2 - \langle us'\rangle^2 - \int s'G(u) + \frac{1}{2}\left\|\mathcal{P}^*s\right\|^2 + \frac{1}{2}\|u\|^2 \quad (3.4) \end{aligned}$$

By (2.5), the nonlinearity can be estimated as follows

$$-\int s'G(u) = -\int s'G_0(u) - \int s'G_1(u) \underset{\text{Schwarz}}{\le} -\int s'G_0(u) + \gamma_1 L^{1/2}\left\|s'\right\| \qquad (3.5)$$

Further, since

$$\left(\alpha + \frac{1}{2}\right)\|u\|^2 = -\frac{1}{2}\|u\|^2 + (\alpha+1)\|u\|^2 \le -\frac{1}{2}\|u\|^2 + \frac{\alpha+1}{c_{\min}}\int G_0(u) \qquad (3.6)$$

the estimate in (3.4) can be continued as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u-s\|^2 \underset{(3.6)-(3.5)}{\le} & -c_0\,\|u\|^2_{H^m} - \langle us'\rangle^2 - \frac{1}{2}\|u\|^2 + \int\left(\frac{\alpha+1}{c_{\min}} - s'\right)G_0(u) \\ & + \frac{1}{2}\left\|\mathcal{P}^*s\right\|^2 + \gamma_1 L^{1/2}\left\|s'\right\| \end{aligned}$$

So far s was arbitrary. Now we select s so that

$$\frac{\alpha+1}{c_{\min}} - s' = b(x) = \beta - s' \qquad (3.7)$$

where $b(x)$ is a function introduced in Lemma 3.1 above.

To guarantee s being periodic we require

$$\int s' = \int \left(\frac{\alpha+1}{c_{\min}} - b(x) \right) \underset{\beta := \frac{\alpha+1}{c_{\min}}}{=} \beta L - B = 0.$$

This implies the parameter B in the definition of $b(x)$ to satisfy $B = \beta L$.

Note that

$$\int ub = \langle ub \rangle = \langle u(\beta - s') \rangle = -\langle us' \rangle.$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u-s\|^2 \quad & \leq \quad -c_0 \|u\|^2_{H^m} - \langle us' \rangle^2 - \frac{1}{2}\|u\|^2 + \int bG(u) \\
& \quad\quad + L^{1/2}\gamma_1 \left\| s' \right\| + \frac{1}{2}\left\| \mathcal{P}^*(s) \right\|^2 \\
& \underset{(3.1)}{\leq} \quad \left(\frac{4C_{\max}}{B} - 1 \right) \langle ub \rangle^2 - c_0 \|u\|^2_{H^m} + B\varepsilon' C_{\max} \int u_x^2 dx \\
& \quad\quad -\frac{1}{2}\|u\|^2 + \frac{1}{2}\left\| \mathcal{P}^*(s) \right\|^2 + \gamma_1 L^{1/2} \left\| s' \right\| \\
& \leq \quad \left(-c_0 + B\varepsilon' C_{\max} \right) \|u\|^2_{H^m} - \frac{1}{2}\|u\|^2 + \frac{1}{2}\left\| \mathcal{P}^*(s) \right\|^2 \\
& \leq \quad -\frac{1}{2}\|u\|^2 + \frac{1}{2}\left\| \mathcal{P}^*(s) \right\|^2 + \gamma_1 L^{1/2} \left\| s' \right\| \\
& \underset{\text{Triangle inequality}}{\leq} \quad -\frac{1}{4}\|u-s\|^2 + \frac{1}{2}\|s\|^2 + \frac{1}{2}\left\| \mathcal{P}^*(s) \right\|^2 + \gamma_1 L^{1/2} \left\| s' \right\|
\end{aligned}$$

Here we assumed $4C_{\max} \leq \beta L$ (i.e., L is sufficiently large) and selected ε' so that $\varepsilon' B C_{\max} = c_0$:

$$\varepsilon' = c_0/(BC_{\max}) = c_0/(\beta L C_{\max}) := c/L \tag{3.8}$$

Next we note that $\|s\|^2$ is estimated as follows

$$s(x) = \int_0^x s' = \int (\beta - b(x)) = \beta x - \int_0^x b(x) := \beta x - g(x)$$

We replace $g(x)$ by its maximum B and use $B = \beta L$ to obtain

$$\|s\|^2 \leq 2\|\beta x\|^2 + 2\|g\|^2 \leq \frac{2}{3}\beta^2 L^3 + 2\beta^2 L^3 = \frac{8}{3}\beta^2 L^3 \tag{3.9}$$

To estimate $\|\mathcal{P}^*(s)\|^2$ we use (3.2) (note that ε in (3.2) is proportional to ε' in (3.8) above)

$$\left\| \mathcal{P}^*(s) \right\|^2 \leq CB^2/\varepsilon^{4m+1} = CL^{4m+3}$$

Similarly, $\left\|s'\right\| \leq CL^{5/2}$. Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-s\|^2 &\leq -\frac{1}{4}\|u-s\|^2+\frac{1}{2}\|s\|^2+\left\|s'\right\|L^{1/2}\gamma_1+\frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2\\ &\leq -\frac{1}{4}\|u-s\|^2+CL^{4m+3}+CL^3 \leq -\frac{1}{4}\|u-s\|^2+CL^{\nu}\end{aligned}$$

where $\nu = 4m+3$. Finally from the Gronwall Lemma we get

$$\begin{aligned}\|u-s\|^2 &\leq \|u_0-s(x)\|^2\exp\left(-\frac{1}{2}t\right)+2CL^{\nu}\left[1-\exp\left(-\frac{1}{2}t\right)\right]\\ &= \left(\|u_0-s(x)\|^2-2CL^{\nu}\right)\exp\left(-\frac{1}{2}t\right)+2CL^{\nu},\end{aligned}$$

showing the exponential approach to the absorbing set. This inequality can be rearranged as

$$\begin{aligned}\|u\| &\leq \|u-s\|+\|s\|\\ &\leq (\|u_0-s\|^2-2CL^{\nu})^{1/2}\exp\left(-\frac{1}{4}t\right)+\sqrt{2C}L^{\nu} \underset{(3.9)}{+} C_sL^{3/2}\\ &\leq (\|u_0-s\|^2-2CL^{\nu})^{1/2}\exp\left(-\frac{1}{4}t\right)+R_a \qquad (3.10)\end{aligned}$$

where

$$R_a := \sqrt{2C}L^{\nu/2}+C_s(L)^{3/2} < C_aL^{(4m+3)/2} \qquad (3.11)$$

and C_a is an absolute constant.

So far the uniform estimate (3.10) has been obtained for L sufficiently large, $L > L_0$. Observe, however, that for smaller values of L one can view a periodic function of period L as periodic with the period, which is a multiple of L, $kL > L_0$ and therefore the estimate holds. ☐

COROLLARY 3.5. *Let* $B_0 = \{||u_0|| \leq R_0\}$ *and* $u(.,t)$ *be a solution of (2.1) with* $u(.,0) = u_0$. *Then*

- *(i) For all* $t > 0$, $||u(.,t)|| \leq R_0 + 2R_a$;
- *(ii) For any* ε *there exist* t_0 *so that* $||u(.,t)|| \leq R_a + \varepsilon$ *for any* $t > t_0$. *Thus* $B_a = \{||u|| \leq R_a + \varepsilon\}$ *is an absorbing set in* $\dot{H}^0_{per}$.

Proof. By (3.10)

$$\begin{aligned}\|u\| &\leq (\|u_0-s(x)\|^2-2CL^{\nu})^{1/2}+R_a\\ &\leq \|u_0-s(x)\|+\sqrt{2C}L^{\nu/2}+R_a\\ &\leq R_0+C_sL^{3/2}+\sqrt{2C}L^{\nu/2}+R_a\\ &\leq R_0+2R_a\end{aligned}$$

On the other hand, since

$$\begin{aligned}\|u\| &\leq (\|u_0 - s(x)\|^2 - 2CL^{\nu})^{1/2}\exp\left(-\frac{1}{4}t\right) + R_a\\ &\leq (R_0 + R_a)\exp\left(-\frac{1}{4}t\right) + R_a\end{aligned}$$

the results follows for $t_0 > 4\ln[(R_0 + R_a)/\varepsilon]$. □

4. Compact attractor

In this section we demonstrate that existence of the absorbing set in $\dot{H}^0_{per}$ proved in the previous section yields the existence of such in any $\dot{H}^s_{per}$ with $s \leq m$, where $2m$ is the order of $\mathcal{P}$. Moreover, we show that there exists a compact attractor for the problem (2.1) in any $\dot{H}^s_{per}$ with $s < m$. A compact attractor is obtained as an ω-limit set of the absorbing set, since its orbit is precompact in $\dot{H}^s_{per}$. As an essential ingredient for the compactness argument, next we prove

THEOREM 4.1. *(Estimate for the m-th derivative). For any $t > 0$, and any $r > 0$,*

$$\left\|\partial^m u(.,t+r)\right\|^2 \leq \frac{R^2}{c_0}\left(\alpha + \frac{1}{r}\right)\exp\left(2M\frac{R^2}{c_0}(\alpha r + 1) + 2\alpha r\right) \tag{4.1}$$

where $R = \|u\,(.,0)\| + 2R_a$ and M is an absolute constant, $M = LM_G^2/(4c_0)$.

Proof. By integrating the estimate (2.8) from t to $t + r$, we see that

$$\int_t^{t+r}\left\|\partial^m u\right\|^2 dt \leq \frac{\alpha}{c_0}\int_t^{t+r}\|u\|^2 + \frac{1}{2c_0}\left(\|u(t)\|^2 + \|u(t+r)\|^2\right) \leq \frac{R^2}{c_0}(\alpha r + 1) \tag{4.2}$$

where $R = R_0 + 2R_a$ is the uniform bound for $\|u\|$, see Corollary 3.5. This estimate will be used below in the application of the Uniform Gronwall Lemma.

To obtain an estimate on $\|\partial^m u(.,t)\|^2$ we multiply the equation by $\overline{\partial^{2m} u}$ and use Parseval's theorem

$$\frac{1}{2}\frac{d}{dt}\left\|\partial^m u\right\|^2 = \int \partial^m u\mathcal{P}\overline{\partial^m u} + (-1)^{-m}\int u_x G'(u)\overline{\partial^{2m} u}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\left\|\partial^m u\right\|^2 \underset{\text{Gårding}}{\leq} -c_0\left\|\partial^m u\right\|^2_{H^m} + \alpha\left\|\partial^m u\right\|^2 + \left|\int u_x G'(u)\partial^{2m} u\right|$$

Next we estimate the nonlinear term

$$\begin{aligned}
\left|\int u_x G'(u)\partial^{2m}u\right| &\leq \int M_G\,|u|\,\left|u_x\partial^{2m}u\right| \\
&\leq M_G \sup|u| \int \left|u_x\partial^{2m}u\right| \\
&\underset{\text{Poincaré}}{\leq} M_G L^{1/2}\,\|u_x\| \int \left|u_x\partial^{2m}u\right| \\
&\underset{\text{Schwarz}}{\leq} M_G L^{1/2}\,\|u_x\|^2\,\|\partial^{2m}u\| \\
&\underset{\text{Young}}{\leq} c_0\left\|\partial^{2m}u\right\|^2 + \frac{LM_G^2}{4c_0}\|u_x\|^4 \\
&\leq c_0\left\|\partial^m u\right\|_{H^m}^2 + \frac{LM_G^2}{4c_0}\|u_x\|^4 \\
&\underset{M:=LM_G^2/(4c_0)}{\leq} c_0\left\|\partial^m u\right\|_{H^m}^2 + M\left\|\partial^m u\right\|^4
\end{aligned}$$

Thus,

$$\frac{d}{dt}\left\|\partial^m u\right\|^2 \leq 2\alpha\left\|\partial^m u\right\|^2 + 2M\left\|\partial^m u\right\|^4$$

Finally the Uniform Gronwall Lemma from [17, p. 91] with $y = \|\partial^m u\|^2$, $g = 2M \|\partial^m u\|^2 + 2\alpha$, and $h = 0$ yields

$$\left\|\partial^m u(., t+r)\right\|^2 \leq \frac{R^2}{c_0}\left(\alpha + \frac{1}{r}\right)\exp\left(2M\frac{R^2}{c_0}(\alpha r + 1) + 2\alpha r\right)$$

□

For reader's convenience we include here the beautiful lemma due to Foias and Prodi [6] which was employed in the proof above:

LEMMA 4.2. *(Uniform Gronwall Lemma, [17, p. 91]) Let* g, h, y *be three positive locally integrable functions on* $]t_0, +\infty[$ *such that*

$$\frac{dy}{dt} \leq gy + h \textit{ for } t \geq t_0,$$

and which satisfy

$$\int_t^{t+r} g \leq a_1, \int_t^{t+r} h(s)ds \leq a_2, \int_t^{t+r} y(s)ds \leq a_3$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1)$$

The following corollary complements the existence of the global absorbing ball in $\dot{H}^0_{per}$ from Theorem 3.4 by establishing stability in all $\dot{H}^s_{per}$ with $s \leq m$.

COROLLARY 4.3. *Let $s \leq m$. Then for any solution u of (2.1)*

$$\limsup_{t \to \infty} ||u(.,t)||_{H^s} \leq R_m \tag{4.3}$$

uniformly for any ball $||u(.,0)||_{H^0} \leq R$. Here R_m is a universal constant. In addition,

$$\frac{1}{t} \int_0^t \left\| \partial^m u \right\|^2 \leq \frac{1}{c_0} \left(\alpha + \frac{1}{t}\right) (R_0 + 2R_a)^2 \tag{4.4}$$

Proof. The proof is obtained from (4.1) by setting r fixed and noting that

$$||u(.,t)||_{H^m} \leq 2^{m/2} ||\partial^m u||$$

and that the s-norm is majorized by the m-norm. Thus

$$R_m = 2^{m/2} \frac{R^2}{c_0} \left(\alpha + \frac{1}{r}\right) \exp\left(2M \frac{R^2}{c_0} (\alpha r + 1) + 2\alpha r\right). \tag{4.5}$$

The estimate in (4.4) is obtained by integrating (2.8) from 0 to t and dividing by t. ☐

Note that while the estimate in (4.3) is exponential in L, the estimate on the integral (4.4) is polynomial.

Now we are ready to establish the compactness result.

THEOREM 4.4. *The initial value problem (2.1) possesses a maximal, connected, compact attractor in $\dot{H}^s_{per}$ for any $s < m$.*

Proof. A compact attractor is obtained as an ω-limit set of the absorbing ball in $\dot{H}^s_{per}$, $\mathcal{B} := B_{R_m+\varepsilon}$, $\mathcal{A} = \cap_{t_0 \geq 0} \overline{\Omega(t_0)\mathcal{B}}$, where $\Omega(t_0)u_0$ is an orbit of the time evolution $S(t)u_0$ with the initial condition u_0: $\Omega(t_0)u_0 = \cup_{t > t_0} S(t)u_0$. For a fixed $r = r_0$ the estimate $\|\partial^m u(.,t+r)\| \leq C$ in (4.1) is uniform for $t \geq r_0$, and all u_0 with $\|u_0\|_{H^s} \leq R$. It holds for any R, including the absorbing ball. Together with the uniform bounds on $\|u(.,t)\|_{H^s}$ from Corollary 4.3 it allows us to use Rellich's theorem and conclude that the orbit $\Omega(t_0)\mathcal{B}$ is precompact in $\dot{H}^s_{per}$. It is known (see e.g. [17, Theorem I.1.1]) that in this case the ω-limit set of $\mathcal{B}$ is a maximal, connected, compact attractor of the flow S. ☐

REMARK 4.5. On the attractor it is not difficult to obtain uniform H^s estimates for any s (not necessarily $s \leq m$) via a boot strap argument. Indeed on the attractor u can be viewed as a solution of the following *linear* parabolic equation

$$u_t = \mathcal{P}(D)u - au_x,$$

where $a(x,t) = G'(u(x,t))$ is a continuous uniformly bounded function, assuming G is sufficiently regular.

5. Dimension of attractor

To obtain the estimate on the Hausdorff dimension of the attractor we study evolution of the infinitesimal volume along the trajectories in the attractor. We demonstrate that for sufficiently large N, the N-dimensional volume decays exponentially. This property combined with the compactness of the attractor and differentiability of the semigroup yields that the Hausdorff dimension of the attractor is no larger than N. In the outline the arguments regarding the Hausdorff dimension of the attractor follow quite closely the ideas presented in [17]. However, we employ a much simpler estimate of the nonlinear term in the trace formula to obtain immediately a bound on the dimension (that is not necessarily optimal in terms of the period L).

Let v be a solution in the attractor $u = v + \varepsilon w$ its perturbation, then for the linear evolution of w we obtain the following problem

$$w_t = -\left[wG'(v)\right]_x + \mathcal{P}w := \mathcal{L}[v]w \tag{5.1}$$

It is easy to see that the linearized problem is well-posed. We are now ready to estimate the evolution of the volume element. To this end we need to estimate the trace of the finite-dimensional projections of the generator of the linear semigroup.

Let $\{w_1, \ldots, w_n\}$ be solutions of the linearized problem with initial values $\xi_1, \ldots, \xi_n \in H^k$. Introduce

$$q_n(t) = \sup_{v(0)\in\mathcal{A}} \sup_{\{\xi_i\}\in H^k} \frac{1}{t}\int_0^t \operatorname{Tr}\left[\mathcal{L}[v(\tau)] \circ Q_n(\tau)\right] d\tau,$$

where $Q_n(\tau)$ is the projector in H^k onto the subspace spanned by $\Xi(\tau) = \{w_1(\tau), \ldots, w_n(\tau)\}$, and denote

$$q_n = \limsup_{t\to\infty} q_n(t).$$

THEOREM 5.6. *For sufficiently large n the n-dimensional volume in the sense of H^k, for integer $k < m$, decays exponentially in time.*

Proof. In order to calculate the trace we choose a basis $\{\phi_1, \ldots, \phi_n\}$ in $\Xi(\tau)$ orthonormal in the sense of H^k.

Then for any $i \leq k$

$$\begin{aligned}\left\langle \partial^i \mathcal{L}[v]\phi_j, \partial^i \phi_j \right\rangle &= \int \partial^i \left(\mathcal{P}\phi_j - \left[\phi_j G'(v)\right]_x\right) \partial^i \phi_j \\ &\underset{\text{Gårding}}{\leq} -c_0 ||\partial^i \phi_j||^2_{H^m} + \alpha ||\partial^i \phi_j||^2 + \left| \int \left(\partial^{i+1}\phi_j\right) \partial^i \left[G'(v)\phi_j\right] \right|,\end{aligned}$$

To estimate $\int \left(\partial^{i+1}\phi_j\right) \partial^i \left[G'(v)\phi_j\right]$ we consider a typical term $(0 \leq \beta \leq i)$

$$\left| \int \left(\partial^{i+1}\phi_j\right) \partial^\beta G'(v) \left(\partial^{i-\beta}\phi_j\right) \right| =$$

$$\leq \sup |\partial^\beta G'(v)| \int |\partial^{i+1}\phi_j|\, |\partial^{i-\beta}\phi_j| \underset{\text{Schwarz}}{\leq} \sup |\partial^\beta G'(v)|\, ||\partial^{i+1}\phi_j||\, ||\partial^{i-\beta}\phi_j||$$

$$\underset{\text{Young}}{\leq} \frac{(i+1)}{2c_0} \left(\sup |\partial^\beta G'(v)|\, ||\partial^{i-\beta}\phi_j||\right)^2 + \frac{c_0}{2(i+1)} ||\partial^{i+1}\phi_j||^2$$

By the Poincaré lemma $\sup |\partial^\beta v| \leq \sqrt{L}||\partial^{\beta+1}v||$. Since v is in the attractor, $||\partial^{\beta+1}v|| \leq R_m$, see (4.3). For the same reason, on the attractor $|v| \leq \sqrt{L}R_m$ and therefore $G(v)$ and all its derivatives up to m on the attractor are bounded by some constant. Thus, the estimate above can be continued

$$\leq K_{\beta,i} ||\partial^{i-\beta}\phi_j||^2 + \frac{c_0}{2(i+1)} ||\partial^{i+1}\phi_j||^2. \tag{5.2}$$

Summation with respect to β yields

$$\left| \int \partial^{i+1}\phi_j \partial^i \left(G'(v)\phi_j\right) \right| \leq K_i ||\phi_j||^2_{H^i} + \frac{c_0}{2} ||\partial^{i+1}\phi_j||^2$$

and therefore

$$\left\langle \partial^i \mathcal{L}\phi_j, \partial^i \phi_j \right\rangle \leq -c_0 ||\partial^i \phi_j||^2_{H^m} + \alpha ||\partial^i \phi_j||^2 + K_i ||\phi_j||^2_{H^i} + \frac{c_0}{2} ||\partial^{i+1}\phi_j||^2$$

Thus, for the sum with respect to i,

$$\begin{aligned}\left\langle \mathcal{L}\phi_j, \phi_j \right\rangle_{H^k} &= \sum_{i=0}^{k} \left\langle \partial^i \mathcal{L}\phi_j, \partial^i \phi_j \right\rangle \\ &\leq -c_0 \sum_{i=0}^{k} ||\partial^i \phi_j||^2_{H^m} + \alpha ||\phi_j||^2_{H^k} + K ||\phi_j||^2_{H^k} + \frac{c_0}{2} ||\phi_j||^2_{H^{k+1}} \\ &\underset{||\phi_j||_{H^k}=1}{\leq} -\frac{c_0}{2} ||\partial^k \phi_j||^2_{H^m} + \alpha + K\end{aligned}$$

Consequently for n modes

$$\mathrm{Tr}\,(\mathcal{L}\circ Q_n) = \sum_{j=1}^{n} \left\langle \mathcal{L}\phi_j, \phi_j\right\rangle_{H^k} \le -\frac{c_0}{2}\sum_{j=1}^{n} ||\partial^k\phi_j||^2_{H^m} + n\,(\alpha + K) \tag{5.3}$$

We note that $-\sum_{j=1}^{n} ||\partial^k\phi_j||^2_{H^m}$ can be estimated via the first n eigenvalues of ∂^{2m+2k}, which in turn are bounded by the first n eigenvalues of ∂^{4m} since $k < m$.

$$-\sum_{j=1}^{n} ||\partial^k\phi_j||^2_{H^m} \le -\frac{1}{4m+1}\left(\frac{n\pi}{L}\right)^{4m+1}$$

where the exponent $4m + 1$ and the factor $(4m + 1)^{-1}$ arise because of summation with respect to the order of the derivative (from 0 to m) in the definition of the norm. Obviously, for

$$n > N := \left(\frac{2}{c_0}\,(4m+1)\,(\alpha + K)\,\frac{L}{\pi}^{4m+1}\right)^{1/(4m)} \tag{5.4}$$

the trace and $q_n(t)$ are negative. Now the theorem follows from the result on Lyapunov exponents [17, Chap. 5]. □

REMARK 5.7. The value of the dimension N in the sense of H^k, $1 \le k < m$, obtained above depends on K, which is expressed through R_m and therefore is exponential in L. If the nonlinearity is purely quadratic, $G(u) = u^2/2$, or a finitely-supported perturbation of such: $G(u) = u^2/2 + G_1(u)$, $\operatorname{supp} G_1 < \infty$, then it is easy to obtain a polynomial estimate on the dimension via (4.4).

Proof. For the simplicity of presentation we sketch the proof for $G(u) = u^2/2$; the modifications for the perturbed case are quite straightforward. The estimate in (5.2) can be replaced by a more subtle estimate

$$\begin{aligned} &\left|\int \partial^{i+1}\phi_j\partial^\beta v\partial^{i-\beta}\phi_j\right| = \\ &\le \sup|\partial^\beta v| \int |\partial^{i+1}\phi_j|\,|\partial^{i-\beta}\phi_j| \underset{\text{Schwarz}}{\le} \sup|\partial^\beta v|\,||\partial^{i+1}\phi_j||\,||\partial^{i-\beta}\phi_j|| \\ &\underset{\text{Young}}{\le} \frac{(i+1)}{2c_0}\left(\sup|\partial^\beta v|\,||\partial^{i-\beta}\phi_j||\right)^2 + \frac{c_0}{2\,(i+1)}||\partial^{i+1}\phi_j||^2 \\ &\underset{\text{Poincaré}}{\le} \frac{(i+1)}{2c_0}L^{2(m-1)}\left(\left\|\partial^m v\right\|\,||\partial^{i-\beta}\phi_j||\right)^2 + \frac{c_0}{2\,(i+1)}||\partial^{i+1}\phi_j||^2 \\ &\underset{||\phi_j||_{H^k}=1}{\le} \frac{(i+1)}{2c_0}L^{2(m-1)}\left\|\partial^m v\right\|^2 + \frac{c_0}{2\,(i+1)}||\partial^{i+1}\phi_j||^2 \end{aligned}$$

Now by integrating the new version of the trace inequality we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t \sum_{j=1}^n \left\langle \mathcal{L}\phi_j, \phi_j\right\rangle dt &\leq -\frac{c_0}{2}\sum_{j=1}^n ||\partial^k \phi_j||_{H^m}^2 + n\alpha + nm^2 L^{2(m-1)}\frac{1}{2c_0 t}\int_0^t \left\|\partial^m u\right\|^2 d\tau \\ &\underset{(4.4)}{\leq} -\frac{c_0}{2}\frac{1}{4m+1}\left(\frac{n\pi}{L}\right)^{4m+1} + n\alpha + \frac{\alpha m^2}{c_0^2} n L^{2(m-1)}\left(3R_a\right)^2\end{aligned}$$

Taking into account the estimate (3.11) for R_a, it produces the following asymptotics for large L

$$n \sim L^{(10m+2)/(4m)}.$$

□

REMARK 5.8. Further, if the dimension is evaluated in the H^0-norm ($k = 0$ in the statement of Theorem 5.6), then a polynomial dimension estimate can be obtained for the general G as defined in (2.4).

Proof. Indeed, by repeating the steps of the proof of Theorem 5.6 we see that

$$\left\langle \mathcal{L}\phi_j, \phi_j\right\rangle \underset{\text{Gårding}}{\leq} -c_0||\phi_j||_{H^m}^2 + \alpha||\phi_j||^2 + \left|\int \partial\phi_j \left(G'(v)\phi_j\right)\right|,$$

The term $\int G'(v)\phi_j\partial\phi_j$ is now estimated as follows

$$\begin{aligned}\left|\int G'(v)\phi_j\partial\phi_j\right| &\underset{|G'|\leq M_G}{\leq} M_G \sup|v| \int |\phi_j|\,|\partial\phi_j| \\ &\underset{\text{Schwarz}}{\leq} M_G \sup|v|\,||\phi_j||\,||\partial\phi_j|| \\ &\underset{\text{Young}}{\leq} \frac{1}{2c_0}\left(M_G \sup|v|\,||\phi_j||\right)^2 + \frac{c_0}{2}||\partial\phi_j||^2 \\ &\underset{\text{Poincare}}{\leq} \frac{1}{2c_0}||M_G\sqrt{L}v_x||^2||\phi_j||^2 + \frac{c_0}{2}||\partial\phi_j||^2 \\ &= \frac{1}{2c_0}M_G^2 L||v_x||^2 + \frac{c_0}{2}||\partial\phi_j||^2 \\ &\leq \frac{1}{2c_0}M_G^2 L^{2m-1}||\partial^m v||^2 + \frac{c_0}{2}||\phi_j||_{H^m}^2\end{aligned}$$

The exponent $2m-1$ arises from the obvious inequality $||v_x||^2 \leq L^{2(m-1)}||\partial^m v||^2$. For n modes

$$\sum_{j=1}^n \left\langle \mathcal{L}\phi_j, \phi_j\right\rangle \leq -\frac{c_0}{2}\sum_{j=1}^n ||\phi_j||_{H^m}^2 + \alpha n + \frac{n}{2c_0}M_G^2 L^{2m-1}||\partial^m v||^2$$

We note that $-\sum_{j=1}^{n} ||\phi_j||_{H^m}^2$ is bounded above by the sum of first n eigenvalues of ∂^{2m},

$$\begin{aligned}\frac{1}{t}\int_0^t \sum_{j=1}^{n} \langle \mathcal{L}\phi_j, \phi_j \rangle dt \;&\le\; -\frac{c_0}{2}(2\pi)^{2m} n^{2m}/L^{2m} + \alpha n + \frac{n}{2c_0} M_G^2 L^{2m-1} \frac{1}{t}\int_0^t ||\partial^m v||^2 d\tau \\ &\underset{(4.4)}{\le} -c_3' n^{2m}/L^{2m} + \alpha n + n M_G^2 \frac{\alpha m^2}{2c_0^2} L^{2(m-1)} (3R_a)^2\end{aligned}$$

The corresponding asymptotics for large L,

$$n \sim M_G^{2/(2m-1)} L^{(8m+1)/(2m-1)}.$$

Note that as the "rate of dissipativity" (the order $2m$ of the operator $\mathcal{P}$) increases, the effect of the particular shape of the nonlinearity, represented here by the M_G-factor becomes less pronounced. ☐

Next, in order to utilize the trace estimate obtained above we need to demonstrate that the nonlinear evolution of the volume is reasonably well approximated by its linear counterpart. The approximation is implied by the differentiability of the evolution semigroup with respect to the initial conditions (cf. [17, Sec. V.3.3]). To be able to carry out the proof of the corresponding theorem, we shall demand the nonlinearity function $G(\eta)$ to be C^3 on an interval of η containing $[-R_a, R_a]$.

THEOREM 5.9. *Let U and W be two orbits in the attractor $U = S(t)U_0$, $W = S(t)W_0$, and $z(t)$ be a solution of the linearized problem (5.1) with the initial condition $z(0) = U_0 - W_0$ (the mapping $z(0) \to z(t)$ is the Frechét differential of $S(t)$ at the point U_0). Then for any t_0,*

$$\|U(t) - W(t) - z(t)\| \le C\ \|U_0 - W_0\|^2, \quad 0 \le t \le t_0,$$

where the constant C depends only on t_0.

Proof. The difference $w = U - W$ solves the following problem

$$\begin{aligned} w_t &= \mathcal{P}(D)w - G'(U)w_x - \left[G'(U) - G'(W)\right] W_x \\ w(x,0) &= U_0(x) - W_0(x) := w_0(x) \end{aligned}$$

As usual we perform the energy estimate:

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \le -c_0 \|w\|_{H^m}^2 + \alpha \|w\|^2 + \left| \int G'(U) w' w \right| + \left| \int W_x G''(\Theta) w^2 \right|$$

where we used the mean-value theorem (Θ is between U and W).

Denote

$$M_k = \max_{|\eta| \le R_a} \left| G^{(k)}(\eta) \right|, \quad k = 1, 2, 3,$$

recall that $G(\eta)$ is assumed to be C^3 on $[-R_a, R_a]$. We see that

$$\begin{aligned}
\left|\int G'(U)w'w\right| + \left|\int W_x G''(\Theta)w^2\right| &\leq M_1 \left|\int w w_x\right| + M_2 \sup|W_x| \int w^2 \\
&\underset{\text{Poincaré}}{\leq} M_1 \left(\frac{1}{2b}\|w\|^2 + \frac{b}{2}\|w_x\|^2\right) \\
&\quad + \sqrt{L} M_2 \|W_{xx}\| \, \|w\|^2 \\
&= \left(\frac{M_1}{2b} + \sqrt{L} M_2 \|W_{xx}\|\right)\|w\|^2 + \frac{b}{2} M_1 \|w_x\|^2
\end{aligned}$$

(Recall that $\|W_{xx}\|$ is uniformly bounded since both U and W belong to the attractor). By choosing b so that $bM_1 = c_0$ we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq -\frac{c_0}{2}\|w\|^2_{H^m} + C\|w\|^2,$$

Thus

$$\|w(.,t)\|^2 \leq \|w_0\|^2 \exp(2Ct)$$

In addition

$$\begin{aligned}
c_0 \int_0^t \|w\|^2_{H^m} &\leq 2C \int_0^t \|w\|^2 + \|w(.,t)\|^2 - \|w_0\|^2 \\
&\leq 2C\|w_0\|^2 \left[\exp(2Ct) - 1\right] \qquad (5.5)\\
&:= C_0 \|U_0 - W_0\|^2
\end{aligned}$$

For the difference $y = w - z$ we have the following problem

$$y_t = \mathcal{P}y - G'(U)y_x + yG''(U)U_x + \left[W_x G''(\Theta) - G''(U)U_x\right] w \qquad (5.6)$$

We multiply the equation in (5.6) by y and integrate by parts to obtain the following inequality for the norm:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y\|^2 &\leq -c_0\|y\|^2_{H^m} + \alpha\|y\|^2 + \left|\int G'(U)y_x y\right| \qquad (5.7)\\
&\quad + \left|\int G''(U)U_x y^2\right| + \left|\int \left[W_x G''(\Theta) - G''(U)U_x\right] wy\right|
\end{aligned}$$

Since

$$\begin{aligned}
\left|W_x G''(\Theta) - G''(U)U_x\right| &= \left|\left[W_x - U_x\right] G''(\Theta) + U_x\left[G''(\Theta) - G''(U)\right]\right| \\
&= \left|w_x G''(\Theta) + U_x G'''(\Theta_1)(\Theta - U)\right| \\
&\leq |w_x| \left|G''(\Theta\right| + \left|U_x G'''(\Theta_1)\right| |w|
\end{aligned}$$

where Θ_1 is between U and Θ, the last integral in (5.7) is estimated as

$$\begin{aligned}\left|\int \left[W_x G''(\Theta) - G''(U)U_x\right] wy\right| &\le \int |w_x w y| \left|G''(\Theta\right| + \int \left|U_x G'''(\Theta_1)\right| \left|w^2 y\right| \\ &\underset{\text{Poincaré}}{\le} \sqrt{L}\, \|w_x\|\, M_2 \int |w_x y| \\ &\quad +L\, \|w_x\|\, \|U_{xx}\|\, M_3 \int |wy| \\ &\underset{\text{Schwartz}}{\le} C\, \|w_x\|^2\, \|y\|\end{aligned}$$

Thus, one can estimate the right hand side of (5.7) as follows:

$$\le -c_0 \|y\|^2_{H^m} + \alpha \|y\|^2 + M_1 \left(\frac{1}{2b}\|y\|^2 + \frac{b}{2}\|y_x\|^2\right) + \sqrt{L} M_2 \|U_{xx}\|\, \|y\|^2 + C\, \|w_x\|^2\, \|y\|$$

We select b so that $M_1 b \le 2c_0$, which yields

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \le C_1 \|y\|^2 + C\, \|w_x\|^2\, \|y\|$$

or

$$\frac{d}{dt}\|y\| \le C_1 \|y\| + C\, \|w_x\|^2 .$$

Now from the Gronwall lemma we obtain

$$\begin{aligned}\|y(.,t)\| &\le e^{C_1 t}\int_0^t C\, \|w_x\|^2 e^{-C_1\tau} d\tau \underset{(5.5)}{\le} C e^{C_1 t}\frac{C_0}{c_0}\|U_0 - W_0\|^2 \\ &\le c\, \|U_0 - W_0\|^2 ,\end{aligned}$$

thus completing the proof of Theorem 5.9. ☐

Finally (see [17, Theorem V.3.1]), the volume estimate Theorem 5.6 and the differentiability Theorem 5.9 lead to the following conclusion:

THEOREM 5.10. *The Hausdorff dimension of the attractor for the problem (2.1) in the class of periodic functions with zero average is finite.*

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