Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 425–430
Applied Mathematics Letters
www.elsevier.com/locate/aml

Stability for a class of nonlinear pseudo-differential equations

Michael Frankel[a], Victor Roytburd[b,*]

[a] Department of Mathematics, Indiana University - Purdue University Indianapolis, Indianapolis, IN 46202-3216, USA
[b] Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180-3590, USA

Received 23 March 2007; accepted 23 March 2007

Abstract

We study a class of nonlinear evolutionary equations generated by a pseudo-differential operator with the elliptic principal symbol and with nonlinearities of the form $G(u_x)$ where $c\eta^2 \le G(\eta) \le C\eta^2$ for large $|\eta|$. We demonstrate existence of a universal absorbing set, and a compact attractor, and show that the attractor is of a finite Hausdorff dimension. The stabilization mechanism is similar to the nonlinear saturation well known for the Kuramoto–Sivashinsky equation.

Keywords: Kuramoto–Sivashinsky equation; Pseudo-differential operators; Dissipativity; Attractors

1. Introduction

The present letter communicates our recent results regarding dissipativity for a broad class of nonlinear equations of the form

$$u_t + G(u_x) = \mathcal{P}(D)u \tag{1}$$

where $\mathcal{P}(D)$ is a linear elliptic pseudo-differential operator of the order $2m$. Eq. (1) may be viewed as a generalization of the Kuramoto–Sivashinsky (KS) equation [10,9,12]. A similar generalization was studied by Nicolaenko et al. [11] under an assumption of the invariance with respect to reflection for both the equation and solutions.

The nonlinearity in (1) is of a more general form than the conventional purely quadratic function. We allow the functions $G(\eta)$ satisfying $c_{\min}^2\eta \le G(\eta) \le C_{\max}^2\eta$, for $|\eta|$ large. We believe however that the latter requirement on the growth of G may be due to our technical approach rather than to the intrinsic nature of the problem.

Equations of type (1) generate a variety of dynamical patterns, due to the interplay between the linearly unstable modes and the nonlinearity. The class of equation (1) includes, for example, the Kawahara equation [14] (a.k.a. the generalized KS equation)

$$u_t + \frac{1}{2}u_x^2 = -u_{xx} + \gamma u_{xxx} - u_{xxxx},$$

describing waves on the surface of a viscous film.

* Corresponding author.
E-mail addresses: mfrankel@math.iupui.edu (M. Frankel), roytbv@rpi.edu (V. Roytburd).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.03.023

Another instance of (1) is the recently introduced nonlocal equation modelling cellular flames:

$$u_t + \frac{1}{2}u_x^2 = u_{xx} + \alpha(I - A)u,$$

where $A := (I - \partial_x^2)^{-1}$. In [2,1] we demonstrated that it generates a cellular–chaotic dynamics. Yet another example is the nonlocal KS equation (see [12,8]) for the flame dynamics in the limit of a small thermal expansion.

By using the technique developed in [2], on the basis of the ideas of [4,7], we are able to remove the reflection invariance restriction of [11]. For the broader class of equation (1) (or rather for its differentiated with respect to x version), we demonstrate existence of an absorbing set, compactness, and finite Hausdorff dimensionality of the attractor.

At the core of the work is the proof of the uniform boundedness of solutions (Theorem 2.4). The proof employs a Gårding inequality for the linear part and a Poincaré type inequality that controls the behavior of the nonlinear part. The bound in H^0 can be extended to the bounds in all H^s with $0 < s \le m$. Stability in all H^s with $0 < s \le m$ automatically guarantees compactness of the dynamical system under consideration in any H^s with $0 \le s < m$.

In the last section of the letter we briefly discuss the Hausdorff dimension of the attractor. The calculations follow a fairly well-established path of the trace estimation (see e.g. [13]); we did not attempt, however, to obtain necessarily the lowest estimate (cf. [6,3] for the KS equation), opting for the transparency of presentation.

A detailed presentation of the results announced in this letter will appear in [5].

2. Absorbing set

It is more convenient (and rather conventional) to study the initial value problem for the Eq. (1) differentiated with respect to x

$$\begin{aligned} &u_t + [G(u)]_x = \mathcal{P}(D)u \\ &u(x,0) = u_0(x), \quad u_0 \in \dot{H}^0_{\text{per}} \end{aligned} \tag{2}$$

in the Sobolev spaces of (L)-periodic functions with zero mean denoted by $\dot{H}^s_{\text{per}}$. In this section we show that solutions of the problem in (2) belong to $L^\infty([0,\infty); \dot{H}^0_{\text{per}})$.

We assume that $\mathcal{P}$ is an elliptic pseudo-differential operator with the symbol $p(\xi)$,

$$\begin{aligned} &p(\xi) = -C\,|\xi|^{2m} + p_1(\xi), \quad m \ge 1 \\ &|p_1(\xi)| \le c\,|\xi|^{\tilde{m}} \quad \text{for } |\xi| \text{ large}, \tilde{m} < 2m. \end{aligned} \tag{3}$$

$\mathcal{P}$ is assumed to preserve the subspace of the zero-mean functions.

Remark 2.1. It is not difficult however to modify all the proofs for *any classical elliptic pseudo-differential operator with periodic coefficients* such that it preserves the space of zero-mean functions. It is also not difficult to extend somewhat the class of acceptable operators to the operators of the form

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$$

where $\mathcal{P}_1$ is an abstract linear operator (not necessarily a pseudo-differential one) in $\dot{H}^0_{\text{per}}$ that satisfies the inequality

$$|\langle \mathcal{P}_1 u, u\rangle| \le \rho\,\|u\|^2_{H^{\tilde{m}}}$$

(cf. [11]). Here $\langle .,. \rangle$ is the H^0 inner product. Again $\mathcal{P}_1$ is assumed to preserve the space of zero-mean functions. Note that because of the specifics of the proof in [11], the operator there is required to be an even function of D alone.

The nonlinearity is assumed to satisfy

$$c_{\min}\eta^2 \le G(\eta) \le C_{\max}\eta^2, \qquad \left|G'(\eta)\right| \le M_G\,|\eta|, \quad \text{for } |\eta| \text{ large} \tag{4}$$

for some constants $c_{\min}$, $C_{\max}$, and M_G. It is obvious that in this case for all η,

$$\begin{aligned} &G(\eta) = G_0(\eta) + G_1(\eta), \quad c_{\min}\eta^2 \le G_0(\eta) \le C_{\max}\eta^2, \\ &\operatorname{supp} G_1 < \infty, \qquad |G_1(\eta)| \le \gamma. \end{aligned} \tag{5}$$

Ellipticity allows us to use a Gårding type inequality: for $\mathcal{P}$ as defined in (3) there exist constants c_0, α such that for $u \in H^m_{\text{per}}(L)$,

$$\operatorname{Re}\langle \mathcal{P}u, u\rangle \le -c_0 \|u\|^2_{H^m} + \alpha \|u\|^2 . \tag{6}$$

For simplicity (to avoid dealing with real parts) we shall assume that $\mathcal{P}$ is a *real operator*.

The following modification of the lemma from [7, Proposition 1] is used in treatment of the nonlinear terms.

Lemma 2.2. *Let $b \in C^\infty$ be the Sobolev's mollifier*

$$b(x) = \begin{cases} 0, & |x| > \varepsilon \\ \dfrac{a_0}{\varepsilon} B \exp\left[-\dfrac{1}{1-(x/\varepsilon)^2}\right], & |x| \le \varepsilon \end{cases}$$

where a_0 is such that the area equals B. Then for any $u \in H^1$

$$\int b(x)G(u(x))\mathrm{d}x \le C_{\max}\left(\frac{4}{B}\langle ub\rangle^2 + B\varepsilon' \int u'(x)^2 \mathrm{d}x\right) \tag{7}$$

where $\varepsilon' = 32\varepsilon\,(a_0/e)^2$, $\langle ub\rangle = \int b(y)u(y)\mathrm{d}y$.

Remark 2.3. It follows easily from the scaling of $b(x)$ that for any real $k \ge 0$

$$\left\|D^k b\right\|^2 \le CB^2/\varepsilon^{2k+1} \tag{8}$$

where for non-integer k the derivative is defined through the Fourier series.

The main result of this section is the following a priori estimate:

Theorem 2.4 (*Existence of an Absorbing Ball*). *For any solution u of (2)*

$$\limsup_{t\to\infty} \|u(.,t)\| \le \begin{cases} C_a L^{2m+3/2} & \text{if } L > L_0 := 4C_{\max}c_{\min}/(\alpha+1) \\ C_a L_0^{2m+3/2} & \text{if } L \le L_0 \end{cases} \tag{9}$$

where C_a is a universal constant; $\|.\|$ is the H^0-norm.

Proof. Let s be a fixed, sufficiently smooth periodic function with 0 mean (to be selected later). Introduce $y(t)$ as a solution of the initial value problem for the ODE:

$$\dot{y}(t) = \int u(x,t)s'(x+y(t))\mathrm{d}x, \qquad y(0) = 0$$

where $u(x,t)$ is a given solution of (2). Consider the curve of the shifts $s(.+\eta)$ of s, $0 \le \eta \le L$, and let

$$\Phi(t) := \frac{1}{2}\int_{-L/2}^{L/2} [u(x,t) - s(x+y(t))]^2 \mathrm{d}x.$$

We compute the derivative of Φ and substitute u_t from (2) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \int u\mathcal{P}(u) + \int sG'(u)u_x - \dot{y}\int us' - \int s\mathcal{P}u.$$

Note that both $\int ss' = 0$ and $\int G'(u)u_x u = 0$. In the calculations we keep the notation s for the shifted function $s(.+y(t))$ and s_0 for $s(.+0)$. Using the definition of $\dot{y}$, we continue the estimate:

$$\underset{\text{Gårding}}{\le} -c_0\|u\|^2_{H^m} + \alpha\|u\|^2 - \langle us'\rangle^2 + \int sG'(u)u_x + \int u\mathcal{P}^* s$$

$$\le -c_0\|u\|^2_{H^m} + \alpha\|u\|^2 - \langle us'\rangle^2 - \int s'G(u) + \frac{1}{2}\left\|\mathcal{P}^* s\right\|^2 + \frac{1}{2}\|u\|^2 . \tag{10}$$

By (5), the nonlinearity can be estimated as follows:

$$-\int s'G(u) = -\int s'G_0(u) - \int s'G_1(u) \underset{\text{Schwarz}}{\leq} -\int s'G_0(u) + \|s'\| \gamma L^{1/2}. \tag{11}$$

Further we have

$$\left(\alpha + \frac{1}{2}\right)\|u\|^2 = -\frac{1}{2}\|u\|^2 + (\alpha+1)\|u\|^2 \leq -\frac{1}{2}\|u\|^2 + \frac{\alpha+1}{c_{\min}}\int G_0(u). \tag{12}$$

Therefore we can continue the estimate in (10) as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-s\|^2 \underset{(11)–(12)}{\leq} -c_0\|u\|^2_{H^m} - \langle us'\rangle^2 - \frac{1}{2}\|u\|^2 + \int\left(\frac{\alpha+1}{c_{\min}} - s'\right)G_0(u) + \frac{1}{2}\left\|\mathcal{P}^*s\right\|^2 + \|s'\| L^{1/2}\gamma.$$

Next we will force the factor in front of G_0 to be exactly the function b defined in the lemma: $(\alpha+1)/c_{\min} - s' := \beta - s' = b(x)$. To guarantee s being periodic one needs $\int s' = 0$, which implies the choice of $B = \beta L$; as an easy consequence, $\langle ub\rangle = -\langle us'\rangle$.

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-s\|^2 &\leq -c_0\|u\|^2_{H^m} - \langle us'\rangle^2 - \frac{1}{2}\|u\|^2 + \int bG(u) + \|s'\| L^{1/2}\gamma + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 \\
&\underset{(7)}{\leq} -\left(1 - \frac{4C_{\max}}{B}\right)\langle ub\rangle^2 - c_0\|u\|^2_{H^m} + B\varepsilon' C_{\max}\int u_x^2 \mathrm{d}x - \frac{1}{2}\|u\|^2 + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 + \|s'\| L^{1/2}\gamma \\
&\leq \left(-c_0 + B\varepsilon' C_{\max}\right)\|u\|^2_{H^m} - \frac{1}{2}\|u\|^2 + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 \leq -\frac{1}{2}\|u\|^2 + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 + \|s'\| L^{1/2}\gamma \\
&\underset{\text{Triangle inequality}}{\leq} -\frac{1}{4}\|u-s\|^2 + \frac{1}{2}\|s\|^2 + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 + \|s'\| L^{1/2}\gamma.
\end{aligned}$$

Here we assumed $4C_{\max} \leq \beta L$ (i.e., L is sufficiently large) and selected ε' so that $\varepsilon' BC_{\max} = c_0$.

Next we note that $s(x) = \int_0^x s' = \int(\beta - b(x)) = \beta x - \int_0^x b(x) := \beta x - g(x)\,\|s\|^2$ and estimate $\|s\|^2$ as follows:

$$\|s\|^2 \leq 2\|\beta x\|^2 + 2\|g\|^2 \leq \frac{2}{3}\beta^2 L^3 + 2\beta^2 L^3 = \frac{8}{3}\beta^2 L^3. \tag{13}$$

By (8)

$$\left\|\mathcal{P}^*(s)\right\|^2 \leq CB^2/\varepsilon^{4m+1} = CL^{4m+3}$$

(note that ε in (8) is proportional to ε' above), and $\|s'\| \leq CL^{5/2}$. Thus

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-s\|^2 &\leq -\frac{1}{4}\|u-s\|^2 + \frac{1}{2}\|s\|^2 + \|s'\| L^{1/2}\gamma + \frac{1}{2}\left\|\mathcal{P}^*(s)\right\|^2 \\
&\leq -\frac{1}{4}\|u-s\|^2 + CL^{4m+3} + CL^3 \leq -\frac{1}{4}\|u-s\|^2 + CL^\nu
\end{aligned}$$

where $\nu = 4m+3$. Finally from the Gronwall Lemma we get

$$\begin{aligned}
\|u-s\|^2 &\leq \|u_0 - s(x)\|^2 \exp\left(-\frac{1}{2}t\right) + 2CL^\nu\left[1 - \exp\left(-\frac{1}{2}t\right)\right] \\
&= \left(\|u_0 - s(x)\|^2 - 2CL^\nu\right)\exp\left(-\frac{1}{2}t\right) + 2CL^\nu,
\end{aligned}$$

showing the exponential approach to the absorbing set. This inequality can be rearranged as

$$\begin{aligned}
\|u\| &\leq \|u-s\| + \|s\| \leq \left(\|u_0 - s\|^2 - 2CL^\nu\right)^{1/2}\exp\left(-\frac{1}{4}t\right) + \sqrt{2C}L^\nu \underset{(13)}{+} C_s L^{3/2} \\
&\leq \left(\|u_0 - s\|^2 - 2CL^\nu\right)^{1/2}\exp\left(-\frac{1}{4}t\right) + R_a
\end{aligned} \tag{14}$$

where $R_a := \sqrt{2C}L^{\nu/2} + C_s(L)^{3/2} < C_a L^{\nu/2}$ and C_a is an absolute constant.

So far the uniform estimate (14) has been obtained for L sufficiently large, $L > L_0$. For smaller values of L one can view a periodic function of period L as periodic with the period $kL > L_0$ and therefore the estimate holds.

Corollary 2.5. *Let $B_0 = \{\|u_0\| \leq R_0\}$ and $u(., t)$ be a solution of (2) with $u(., 0) = u_0$. Then*
(i) *For all $t > 0$, $\|u(., t)\| \leq R_0 + 2R_a$.* (ii) *For any ε there exists t_0 so that $\|u(., t)\| \leq R_a + \varepsilon$ for any $t > t_0$. Thus $B_a = \{\|u\| \leq R_a + \varepsilon\}$ is an absorbing set in $\dot{H}^0_{\text{per}}$.*

3. Compact attractor and its dimension

In this section we briefly outline further properties of the dynamics that can be derived from the existence of the absorbing set in $\dot{H}^0_{\text{per}}$. The proofs are omitted and will appear elsewhere [5]. First, one can show that an absorbing set exists in any $\dot{H}^s_{\text{per}}$ with $s \leq m$. This implies that for the problem (2) there exists a compact attractor in any $\dot{H}^s_{\text{per}}$ with $s < m$, which is obtained as an ω-limit of the absorbing set, since its orbit is precompact in $\dot{H}^s_{\text{per}}$. The latter is based upon the following a priori estimate.

Theorem 3.1. *(Estimate for the m-th Derivative). For any $t > 0$, and any $r > 0$,*

$$\left\|\partial^m u(., t + r)\right\|^2 \leq \frac{R^2}{c_0}\left(\alpha + \frac{1}{r}\right)\exp\left(2M\frac{R^2}{c_0}(\alpha r + 1) + 2\alpha r\right) \tag{15}$$

where $R = \|u\,(., 0)\| + 2R_a$ and M is an absolute constant, $M = LM_G^2/(4c_0)$. Consequently,

$$\limsup_{t\to\infty} \|u(., t)\|_{H^s} \leq R_m \tag{16}$$

uniformly for any ball $\|u(., 0)\|_{H^0} \leq R$, $s \leq m$. Here R_m is a universal constant.

Through a rather standard argument the estimates in (9) and (16) yield:

Theorem 3.2. *The initial value problem (2) possesses a maximal, connected, compact attractor in $\dot{H}^s_{\text{per}}$ for any $s < m$.*

To obtain the estimate on the Hausdorff dimension of the attractor we study evolution of the infinitesimal volume along the trajectories in the attractor. We demonstrate that for a sufficiently large N, the N-dimensional volume decays exponentially. In the outline the argument regarding the Hausdorff dimension of the attractor follows quite closely the ideas presented in [13].

Theorem 3.3. *For the linearized evolution, the n-dimensional volume in the sense of $H^k, k \leq m - 1$, decays exponentially in time for sufficiently large n.*

Remark 3.4. The value of the dimension N in the sense of H^k, $1 \leq k \leq m-1$, obtained above depends on K, which is expressed through R_m and therefore is exponential in L. If the nonlinearity is purely quadratic, $G(u) = u^2/2$, or a finitely supported perturbation of such: $G(u) = u^2/2 + G_1(u)$, $\operatorname{supp} G_1 < \infty$, then it is easy to obtain a polynomial estimate.

Remark 3.5. Further, if the dimension is evaluated in the H^0-norm ($k = 0$ in the statement of Theorem 3.3), then a polynomial dimension estimate can be obtained for the general G defined in (4). We have the corresponding asymptotics for large L:

$$n \sim M_G^{2/(2m-1)} L^{(8m+1)/(2m-1)}.$$

Note that as the strength of dissipativity (the order $2m$ of the operator $\mathcal{P}$) increases, the effect of the particular shape of the nonlinearity, represented here by the M_G-factor, becomes less pronounced.

The trace estimate described above together with the rather routine differentiability of the evolution semigroup (cf. [13, Sec. V.3.3]) allows us to state the following

Theorem 3.6. *The Hausdorff dimension of the attractor for the problem (2) is finite.*

Acknowledgement

The first author is supported in part by the National Science Foundation Grant DMS-0207308.

References

[1] C.-M. Brauner, M. Frankel, J. Hulshof, G.I. Sivashinsky, Weakly nonlinear asymptotics of the κ–θ model of cellular flames: The QS equation, Interfaces Free Bound 7 (2005) 131–146.
[2] C.-M. Brauner, M. Frankel, J. Hulshof, V. Roytburd, Stability and attractors for quasi-steady equation of cellular flames, Interfaces Free Bound 8 (2006) 301–316.
[3] J.C. Bronski, T. Gambill, Uncertainty estimates and L_2 bounds for the Kuramoto–Sivashinsky equation, Nonlinearity 19 (2006) 2023–2039.
[4] P. Collet, J.-P. Eckmann, H. Epstein, J. Stubbe, A global attracting set for the Kuramoto–Sivashinsky equation, Comm. Math. Phys. 152 (1993) 203–214.
[5] M. Frankel, V. Roytburd, Dissipative dynamics for a class of nonlinear pseudo-differential equations, J. Evol. Equ. (to be submitted).
[6] L. Giacomelli, F. Otto, New bounds for the Kuramoto–Sivashinsky equation, Comm. Pure Appl. Math. 58 (2005) 297–318.
[7] J. Goodman, Stability of the Kuramoto–Sivashinsky and related systems, Comm. Pure Appl. Math. 47 (1994) 293–306.
[8] D. Hilhorst, L.A. Peletier, A.I. Rotariu, G.I. Sivashinsky, Global attractor and inertial sets for a non-local Kuramoto–Sivashinsky equation, Discrete Contin. Dyn. Syst. Ser. A 10 (2004) 557–580.
[9] Y. Kuramoto, Diffusion induced chaos in reactions systems, Progr. Theoret. Phys. Suppl. 64 (1978) 346–367.
[10] A.A. Nepomnyashchy, Stability of wave regimes in a film flowing down on inclined plane, Fluid Dynam. 9 (1974) 354–359.
[11] B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of the Kuramoto–Sivashinsky equations: Nonlinear stability and attractors, Physica D 16 (1985) 155–183.
[12] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames, part I. Derivation of basic equations, Acta Astronaut. 4 (1977) 1177–1206.
[13] R. Temam, Infinite-Dimension Dynamical Systems in Mechanics and Physics, 2nd ed., in: Appl. Math. Sci., vol. 68, Springer, 1997.
[14] J. Topper, T. Kawahara, Approximate equations for long nonlinear waves on a viscous film, J. Phys. Soc. Japan 44 (1978) 663–666.