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# Wrinkling and Fractalization of Curves in an Interface Model of Diffusionally Unstable Flames

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**Abstract**—An equation of interface evolution modeling flame fronts and solid-liquid interfaces is studied numerically. The solutions exhibit two distinct types of behavior. In the first scenario, the curves develop chaotic cellular pattern and accelerate while embedding is sustained. In the second, self-intersections occur and equation gives birth to fractal-like structures with exponentially increasing length and only linearly increasing diameter, which results in dense covering of the plane.

## 1. INTRODUCTION

The equation of curve dynamics in the plane:

$$V_n(s,t) = V + \alpha\kappa + \beta\left(\frac{\partial^2\kappa}{\partial s^2}\right) \tag{1.1}$$

(where $\mathbf{r}(s,t)$ is the position of the curve, $t$ is time, $s$ is the arc-length, $\mathbf{n}$ is the normal, $V_n = \left(\frac{\partial \mathbf{r}}{\partial t}\right) \circ \mathbf{n}$ is the normal velocity, $\kappa$ is the curvature, $V$, $\alpha$, $\beta$ are constant coefficients) was derived as a reduction of the thermal-diffusional model of combustion [1,2] and the nonequilibrium rapid solidifcation of a supercooled pure melt [3].

It was suggested in [1–4] that due to intrinsic instability, equation (1.1) is capable of generating cellular structure and self-chaotization of the interfaces. One of the reasons for such a claim is due to the fact that, for a weakly perturbed plane, equation (1.1) can be asymptotically reduced to the Kuramoto-Sivashinsky equation [5]:

$$y_t + \frac{1}{2}y_x^2 = y_{xx} + y_{xxxx} = 0 \tag{1.2}$$

which is known to exhibit turbulent solutions. In the present note, we briefly present the results of numerical simulation of equation (1.1) that not only confirm its ability to imitate cellular interfaces but also show a rather unexpected and fascinating behavior and allow us to gain some insight into the mathematical nature of the equation.

Indeed, in a certain parameter range, an initially circular expanding curve develops a cellular structure which later becomes turbulent while the general near circular shape is retained. However, there is a region in the parameter space $(V, \alpha, \beta)$ where the picture is drastically different as

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the cellular structure is no longer sustained and leads to self-intersections. Although the physical interpretation of equation (1.1) as a flame sheet is lost, nothing in the equation itself precludes self-intersecting evolution as long as the curve remains sufficiently smooth. As a result, the solution develops a fractal-like structure whose immersion into the plane becomes increasingly dense.

The most important observations that one can make here are as follows: the curve *always* remains smooth while its maximum curvature is *uniformly bounded*, the diameter of the "cloud" continues to increase *linearly*, but the total length of the curve increases *exponentially*, and the curve tends to *evolve* into a fractal of full measure (Hausdorff dimension two). Thus, equation (1.1) is capable of generating two types of behavior: sustained cellular front with roughly linear growth of length or dynamic *self-fractalization*. At the same time, the "initial value" problem in either case is *globally well posed* for sufficiently smooth initial configurations.

We intend to present a substantially more detailed description and discussion of these observations elsewhere in the near future.

## 2. NUMERICAL METHOD AND RESULTS

In the numerical experiments described below we used a very straightforward "marker" technique. The equation was left in geometrcal form, the derivatives with respect to the arc-length were evaluated through finite differences, and the discrete curve representation by a polygon was homogenized with respect to the arc-length to preserve a fixed equal spacing after every time step via a polynomial interpolation.

The explicit finite-difference approximation combined with our hardware limitations (PC) did not allow us to use very fine grids, since the well-known stability condition requires extremely small time steps due to the high order principal term in the "locally parabolic" equation (1.1). We should remark, however, that the exponential growths of the total length, which is probably the most fascinating feature of the dynamics, would create similar difficulties even with a supercomputer.

The algorithm was verified via both time and space step refinement. The results are always qualitatively consistent and quantitatively so for modern times. We also tested our scheme in combination with a nonlocal term modeling hydrodynamical instability in flames [6], and it seemed to produce reliable results consistent with physical predictions whenever such verification was possible [7]. It would be interesting, however, to carry out numerical simulation of equation (1.1) using a more "state of the art" method.

### Weakly Turbulent Fronts

Figure 1 shows spontaneous formation of cellular structure from an initially near circular configuration which is then sustained for the entire period of the experiment. After wrinkling, the velocity of the surface undergoes a jump, but the diameter of the set continues to increase linearly with time. We observe the cells moving and splitting along the front in a chaotic manner while the general outline remains almost circular.

Thus, equation (1.1) does indeed describe evolution of the diffusionally unstable flame fronts, and, consequently, the increase in the propagation velocity which is in full agreement with the physics of the phenomenon. We observe also that the rate of growth of the total length of the curve increases due to formation of the cellular-chaotic structure and is stabilized (on the average) on a new level. At the same time, the maximal curvature, after a substantial increase due to self-chaotization of the curve, remains almost constant and, more importantly, *uniformly bounded in time.* This important feature is preserved for all scenarios discussed below and for all possible intial configurations.

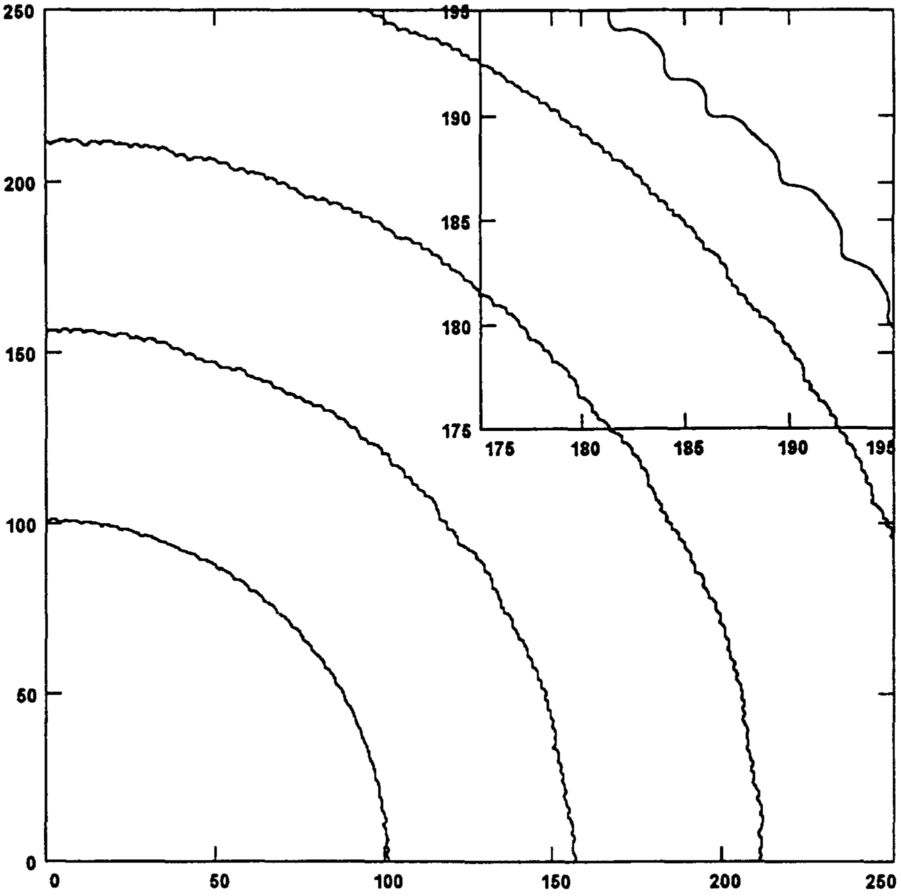

Figure 1. Spontaneous wrinkling and self-chaotization of initially circular contour, $V = 1$, $\alpha = .4$, $\beta = -.1$ (a). Curve position at $t = 100, 150, 200, 250$ (in the window: a magnified segment).

## Self-Intersections and Further Evolution

As we decrease the value of $V$, the initial evolution (Figures 2a and 2b) is simular to that discussed above. However, further evolution leads to collisions between the cells and self-intersection (Figures 2c and 2g). This suggests a possibility that a wrinkled flame front may leave in its wake small droplets of unburned matter, which either continue to burn separately or dissipate. One should be aware, however, that the direct physical interpretation of equation (1.1) as a flame front is no longer applicable since certain conditions under which it was derived cannot be met.

Note, however, that nothing precludes evolution with self-intersections as long as local geometrical characteristics of the curve are well-defined. Thus, in spite of the loss of specific physical interpretation, we continue the experiment, since observation of further evolution may lead to a better understanding of the mathematical nature of equation (1.1). We notice that the bubbles develop into "cascading" substructures (Figure 2h) that can be visualized as multiple twisting and folding a wire loop which, although becoming increasingly complex, does not "tighten up" or, in other words, *the maximal curvature remains bounded.*

We also observe that the general external outline of the curve remains quasi-circular and continues to expand with *constant* mean velocity (Figures 2e and 2f), while the total length of the curve begins to increase *exponentially* right from the moment of formation of bubbles. Under these conditions, taking into account the homogeneous and isotropic nature of equation (1.1), it is already quite natural that the curve tends to cover the whole plane. In order to get some idea of its distribution, one can examine, for instance, the behavior of the distance from the origin, $|\mathbf{r}(s,t)|_{t=\text{const}}$ for some well-developed configuration and notice multiplicity of scales and a degree

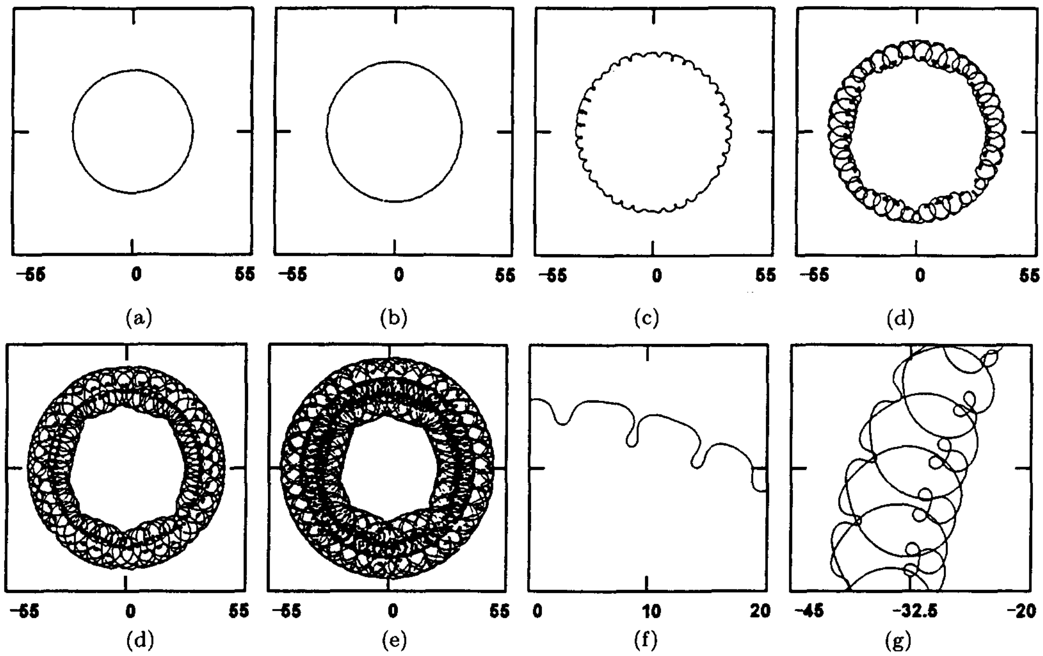

Figure 2. Formation of bubbles and self-fractalization, $V = .4$, $\alpha = .4$, $\beta = -.1$. (a)–(f): Configurations for $t = 20, 30, 40, 50, 60, 70$. (g)–(h): Magnified segments at $t = 40, 50$.

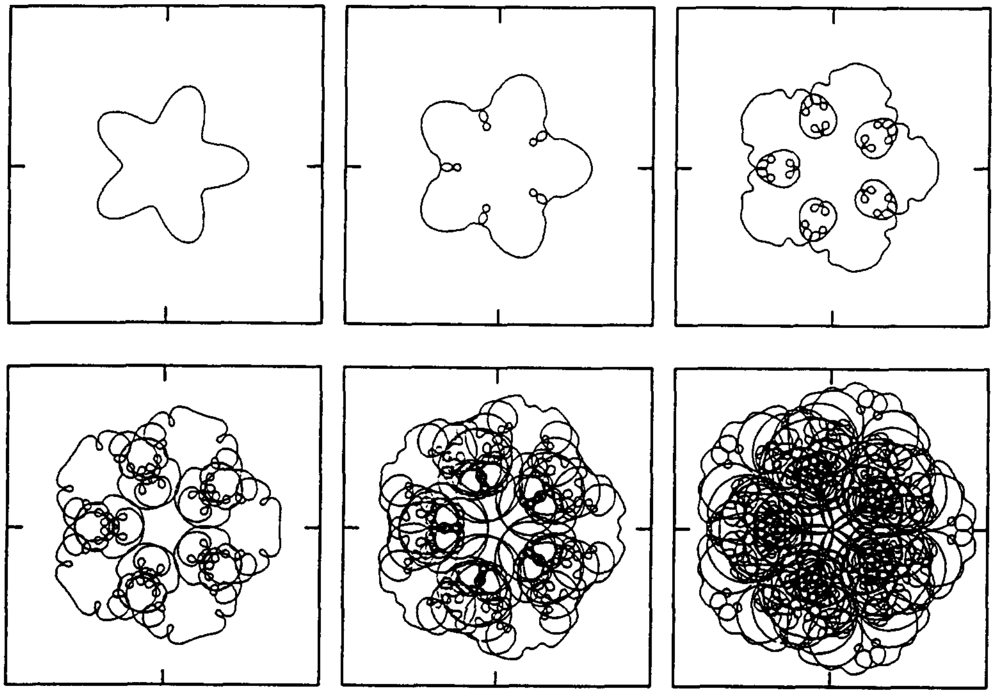

Figure 3. Evolution with 6-fold symmetry, $V = .2$, $\alpha = .4$, $\beta = -.2$. Configurations for $t = 0, 15, 30, 45, 60, 75$.

of self-similarity which suggests a fractal-like behavior (see remark on self-fractalizing families below).

## An Ornament with m-Fold Symmetry

One more interesting observation should be added. We run our experiments with an m-fold rotation and reflection symmetry of the intial configuration. The main features of evolution described above, namely uniformly bounded curvature, exponential growth of length, linear growth of the diameter, dense covering, multiple scales, etc., remain intact but, additionally, the symmetry is also preserved, generating a variety of self-transforming ornaments (Figure 3). This observation suggests the existence of a large class of symmetrical solutions of equation (1.1) due, again, to its homogeneous and isotropic nature.

It turns out that (any) symmetry is sustained, but only for a finite time, except for the case $V = 0$ when it seems to be preserved indefinitely. As for the case $V = 0$ (see footnote)[1]: if our observations are correct, it raises a rather interesting question concerning stability of the symmetric solutions.

## 3. "FRACTAL DIMENSION" OF SOLUTIONS

The term fractal dimension has been used rather freely in recent literature, at times in situations (similar to ours) where it is not directly applicable. Indeed, the well-known definitions deal with the sets that do not evolve. On the other hand, the fractal dimension of a smooth evolving curve, which is a solution of equation (1.1) is equal to 1 at any fixed time. The usual intuitive argument, that it becomes a fractal as $t \to \infty$ is rather naïve, since the solution does not converge to any limit set in any reasonable sense.

In other words, the process of complexification, abusing the term, that we observe in the experiments does not mean attraction to any final state. Therefore, we should clearly realize that we are trying to describe a sort of *dynamic self-fractalization of the family* of curves. It occurs as a result of involvement of increasingly larger scales, that is rather opposite to the conventional fractals. In this context, one should modify the definition of Hausdorff dimension, for instance, taking into account the presence of the smallest scale (minimum of the radius of curvature), when calculating the relative measure of the covering of the curve, and pass to the limit as $t \to \infty$. If the limit exists it should be interpreted as an asymptotic Hausdorff dimension of the family.

We shall attempt to give a rigorous definition of the asymptotic dimension elsewhere. Meanwhile, the reader should regard our claim that the curve seems to become a fractal as the self-fractalizing property in the spirit of the above discussion. The conclusions to be drawn from the

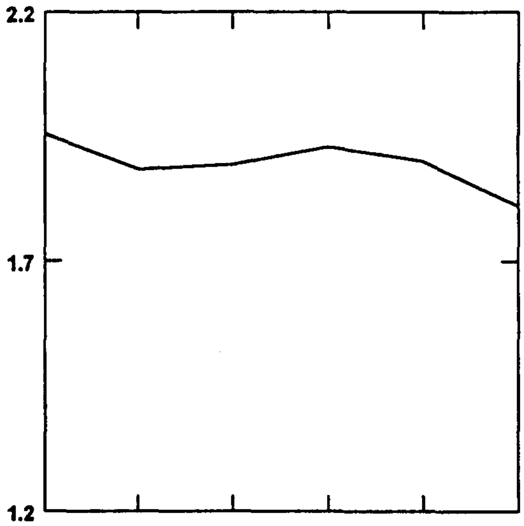

Figure 4. The slope of the graph of ln $N(c)$ vs ln $c$ for $t = 105$, where $N(c)$ is the number of $c$-size squares covering the curve, $c = D(t)2^{-k}$, $k = 1, \dots, 7$. Horizontal axis: $k$, vertical axis: $[\log_2 N(k-1)]/2$.

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[1] Equation (1.1) with $V = 0$ and some additional nonlinear terms was suggested as a phenomenological model of dendrite growth [3].

numerical estimate are not only that the *asymptotic* Hausdorff dimension is greater than unity (above, say, 1.8), but also, that quite certainly it tends to converge *to two* (see Figure 4).

In other words, self-fractalization does indeed occur, and, moreover, the curve tends to occupy full measure in the plane. These experimental observations have been confirmed for a considerable number of various initial configurations and combinations of the parameters $V$, $\alpha$, and $\beta$ in the range where the exponential growth of the length occurs. We should also add that we attempted to estimate the correlational dimension of the family (see e.g. [8]), which also shows the tendency of approaching two.

## REFERENCES

1. M.L. Frankel and G.I. Sivashinsky, On the nonlinear thermal diffusive theory of curved flames, *J. Physique* **48**, 25–28, (1987).
2. M.L. Frankel and G.I Sivashinsky, On the equation of curved flame front, *Physica D* **30**, 28–42, (1988).
3. M.L. Frankel, On the nonlinear evolution of solid-liquid interface, *Phys. Let. A*, 57–60, (1988).
4. R.C. Brower, D.A. Kessler, J. Koplik and H. Levine, Geometrical models of interface evolution, *Phys. Rev. A* **29**, 13–35, (1984).
5. G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames. Part 1. Derivation of basic equations, *Acta Astronautica* **34**, 1177–1206, (1977).
6. M.L. Frankel, An equation of surface dynamics modeling flame fronts as density discontinuities in potential flows, *Phys. Fluids A* **2** (10), 1879–1883, (1990).
7. M.L. Frankel and G.I. Sivashinsky, On fingering instability in non-adiabatic low Lewis number flames, *Phys. Rev. E*, (1995) (to appear).
8. Y.A. Pesin, On rigorous mathematical definition of correlational dimension and generalized spectrum for dimension, *J. Stat. Phys* **71** (3/4), 529–547, (1993).
9. D. Ruelle, *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, (1989).